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Quantum Phase Crossover in the Spherical Mean-field plus $Q\cdot Q$ and Pairing Model within a Single-j Shell

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Abstract: The analysis of the quantum phase crossover behavior in the spherical shell model mean-field plus the geometric quadrupole-quadrupole ($Q\cdot Q$) and standard pairing model within a single-j shell is reported. Several quantities, such as low-lying excitation energies, the overlaps of excited states, ratios of some B(E2) and electric quadrupole moments of some low-lying states as functions of the control parameter of the model in a $j=15/2$ shell are calculated. The results show that there are noticeable changes in the crossover region of the rotational-like to the pair-excitation phase transition, such as $B(\mathrm{E2};4_1\rightarrow 2_1)/B(\mathrm{E2};2_1\rightarrow 0_g)$ and $B(\mathrm{E2};4_2\rightarrow 2_1)/B(\mathrm{E2};2_1\rightarrow 0_g)$, especially in the half-filling case. Though the low-lying excitation energies generated from the geometric quadrupole-quadrupole interaction not satisfy the pattern of a rotational spectrum when j is small, these energies follow the pattern of a rotational spectrum when j is sufficiently large.

Key words: shell model; quadrupole-quadrupole interaction; single-j shell

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1 Introduction

The quantum phase transition in the large-N limit or the quantum phase crossover (QPC) in finite quantum systems is of great interests in many areas of physics[1–2]. In a finite quantum many-body system, the crossover occurs when the interaction strengthes or called control parameters reach a critical point of different phases described by distinct types of ground or excited states of the system[1, 3]. In atomic nuclei, the quantum phase transition has been studied extensively in either the interacting Boson model (IBM) or the Bohr-Mottelson model (BMM)[4–5]. It is now commonly accepted that the three limiting cases of the IBM correspond to three different geometric shapes of nuclei. More interesting scenarios occur when a system is in between two different phases, in which case a quantum phase transition occurs at the corresponding critical point with the distinct symmetry[6–7], *e. g.*, the critical point of the spherical to γ-unstable shape phase transition with the $E(5)$ symmetry[8], the critical point of the spherical to axially deformed shape phase transition with the $X(5)$ symmetry[9], and the critical point of the prolate to oblate shape phase transition with the $Z(5)$ symmetry[10], *etc.*, in the BMM, which have been widely confirmed.

On the other hand, it is widely accepted that the spherical shell model is fundamental in describing low-lying excitation spectra of nuclei when the most important residual interactions, such as the $Q\cdot Q$ and pairing interactions, are taken into consideration[11–14]. In Refs. [15–16], the Hamiltonian with the Elliott's dynamic $Q\cdot Q$ and the standard pairing interaction restricted to a single spherical harmonic oscillator shell was studied, which sheds light on the nature of the QPC of the model. The analysis of the QPCs in the shell model restricted in the SD-pair subspace was also made[17–18].

In this paper, the QPC behaviors of the shell model mean-field plus the geometric $Q\cdot Q$ and the standard pairing interaction within a single-j shell will be shown, of which the application of the model for typi-

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cal nuclei in the rotational-pairing crossover region was provided in our recent work[19].

2 The model

The Hamiltonian of the spherical shell model mean-field plus the geometric $Q\cdot Q$ and the standard pairing interaction within a single-j shell is given by[19]

$$\hat{H}=\epsilon_j\hat{n}_j-GS_+S_--\kappa\widetilde{Q}\cdot\widetilde{Q}, \tag{1}$$

where ϵ_j is the single-particle energy generated from the spherical shell model, G and κ are the pairing and $Q\cdot Q$ interaction strength, respectively, $S_+=\sum_{m>0}(-1)^{j-m}a^\dagger_{jm}a^\dagger_{j-m}$ ($S_-=(S_+)^\dagger$) are pair creation (annihilation) operator, in which $a^\dagger_{jm}$ is the creation operator of a particle in the single-j shell, and $\widetilde{Q}_q=Q_q/r_0^2$ with $r_0^2=\hbar/M_n\omega_0=1.012A^{\frac{1}{3}}$ fm^2, in which M_n is the mass of a nucleon, ω_0 is the frequency of the harmonic oscillator potential. In the spherical shell model basis for a single-j shell, the geometric Q_q can be written as

$$Q_q=\frac{1}{\sqrt{5}}\Big\langle Nl|r^2|Nl\Big\rangle\Big\langle\Big(l\frac{1}{2}\Big)j\Big\|Y^2\Big\|\Big(l\frac{1}{2}\Big)j\Big\rangle(a^\dagger_j\times\tilde{a}_j)^2_q, \tag{2}$$

where

$$\langle Nl|r^2|Nl\rangle=\Big(N+\frac{3}{2}\Big)r_0^2, \tag{3}$$

$$\Big\langle\Big(l\frac{1}{2}\Big)j\Big\|Y^2\Big\|\Big(l\frac{1}{2}\Big)j\Big\rangle=(-1)^{j-\frac{1}{2}}\frac{2j+1}{2\sqrt{\pi}}\Big\langle j\frac{1}{2}j-\frac{1}{2}|20\Big\rangle, \tag{4}$$

and

$$(a^\dagger_j\times\tilde{a}_j)^2_q=\sum_{m_1m_2}\langle jm_1,jm_2|2q\rangle a^\dagger_{jm_1}\tilde{a}_{jm_2}, \tag{5}$$

in which $\tilde{a}_{jm}=(-)^{j+m}a_{j\,-m}$, and $\langle jm_1,jm_2|2q\rangle$ is the Clebsch-Gordan coefficient.

In order to investigate the QPC in this model, we set $G=cx$ and $\kappa=c(1-x)\xi$, in which c is a scale factor, ξ is a dimensionless constant, and x can be taken a value in the closed interval $[0,1]$. For a given number of particles n, up to a constant, the model Hamiltonian is rewritten as

$$\hat{H}'=-c\Big(xS_+S_-+(1-x)\xi\widetilde{Q}\cdot\widetilde{Q}\Big). \tag{6}$$

Thus, x serves as the control parameter of the model, while the parameters ξ is chosen with $0<\xi<1$ in order to keep the critical (or crossover) point not far from $x\sim0.5$ according to the value of j.

The Hamiltonian (6) is diagonalized in the U$(2j+1)\supset$ Sp$(2j+1)\supset$ O(3) basis, of which the basis vectors are denoted as $|[1^n]\langle1^\nu\rangle\alpha JM\rangle$, where n is the number of valence particles in the single-j shell, ν is the seniority quantum number, J and M are the quantum number of the total angular momentum and that of its third component, respectively, and α is the additional quantum number needed to distinguish different state with other quantum numbers the same because Sp$(2j+1)\downarrow$ O(3) is not branching multiplicity-free. The branching multiplicity of J in the reduction Sp$(2j+1)\downarrow$ O(3) for a given ν can be obtained by using the method shown in Refs. [19–20]. Hence, the eigenstates of Eq. (6) are expressed as

$$|n,\zeta,JM\rangle=\sum_{\nu\alpha}C^{(\zeta)}_{\nu\alpha J}|[1^n]\langle1^\nu\rangle\alpha JM\rangle, \tag{7}$$

where ζ labels the ζ-th excited state with the total angular momentum J, and $C^{(\zeta)}_{\nu\alpha J}$ is the corresponding expansion coefficient, namely,

$$\hat{H}'|n,\zeta,JM\rangle=E(J_\zeta)|n,\zeta,JM\rangle, \tag{8}$$

where $E(J_\zeta)$ is the corresponding excitation energy.

3 The QPC behaviors

In order to investigate the QPC in the model, we take j=15/2 in the N=7 major shell as an example, which is the largest single-j shell suitable to describe some experimentally reachable heavy nuclei. The present model concentrates on an ideal system, so we set ξ=0.22 in order to keep the critical (or crossover) point not far from $x\sim0.5$ according to the value of j. To show how the low-lying excitation spectra change as functions of the control parameter x and the total number of particles n, some low-lying excitation spectrum as functions of x for the system with the number of particles n=8 corresponding to the half-filling is shown in Fig. 1.

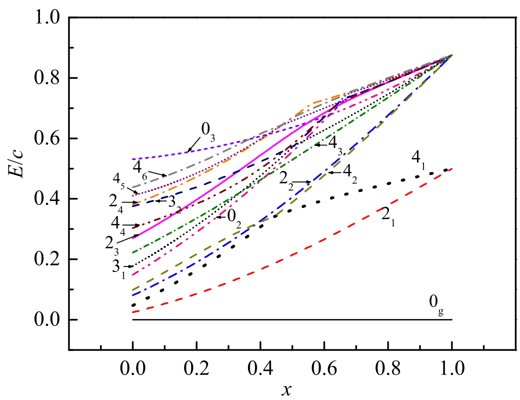

Fig. 1 (color online) Some low-lying excitation spectrum with $j=\frac{15}{2}$ across the transitional region for n=8, where x=0 corresponds to the rotation-like phase and x=1 to the pair-excitation phase. Taken from Ref. [19].

It can be observed from Fig. 1 that the ratios $R_{4/2}$ for $x=0$ are always smaller than the corresponding values of the typical rotational spectrum. The system with $x=0$ is called in the rotation-like phase. With the increasing of x from 0 to 1, the system is driven from the rotation-like phase toward the crossover point with $x\sim 0.5$ because sharp change in level energy never happens due to the fact that the number of particles is always small and finite, and then is driven away from the crossover point toward the pair-excitation phase. The level pattern when $x<0.5$ or >0.5 are noticeably different from that when $x\sim 0.5$. It can be clearly observed from Fig. 1 that there are many level-crossings when $x<0.5$ or $x>0.5$, which are typical for a quasi-integrable system because the model in such cases is either driven by the $Q\cdot Q$ or driven by the pairing interaction. The situation for $x\sim 0.5$, however, is quite different, in which no level-crossing happens, but, instead, there is an obvious level-repulsion for the 4_1 and 4_2 levels around $x=0.5$ as clearly shown in Fig. 1.

In addition, as pointed out previously, within a single finite j shell, spectrum of the geometric quadrupole-quadrupole interaction adopted in this work does not follow the rotational pattern as that generated by the Elliott's dynamic quadrupole-quadrupole interaction, especially in the higher J levels. In order to show the level pattern of the geometric quadrupole-quadrupole interaction, we diagonalize the $-\kappa\widetilde{Q}\cdot\widetilde{Q}$ term in the $n=2$ subspace, of which the eigen-energy $E(J)$ for an allowed total angular momentum J is given by

$$E(J)=-h_0\frac{(2j+1)^2}{\pi}\left\langle j\frac{1}{2}j-\frac{1}{2}\middle|20\right\rangle^2\times\sum_{J'}(2J'+1)\begin{Bmatrix} j & j & 2\\ J & J' & j\end{Bmatrix}^2, \tag{9}$$

where $h_0=\kappa(N+\frac{3}{2})^2$. The low-lying excitation spectra for small j cases generally are away from the rotational pattern. When j is sufficiently large, the low-lying excitation spectra gradually follow the rotational pattern. We have calculated $E(J)$ with $J=0,2,\cdots,10$ from $j=463/2$ to $j=475/2$, of which the results are shown in Fig. 2.

As clearly shown in Fig. 2, for a sufficiently large j shell, which may be regarded as a single-j approximation to the actual situation in nuclear system with multi-j shells, the low-lying excitation spectra of the $-\kappa\widetilde{Q}\cdot\widetilde{Q}$ term indeed gradually follow the rotational pattern. Our conclusion is that only low-lying excitation spectra generated from the geometric quadrupole-quadrupole interaction follow the pattern of a rotational spectrum in a single-j shell when j is sufficiently large.

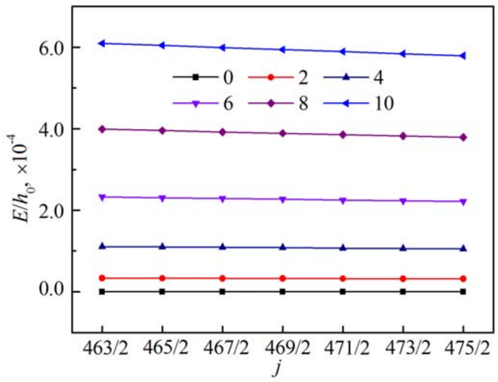

Fig. 2　(color online) Some low-lying excitation spectra with total angular momentum $J=0,2,\cdots,10$ generated from the $-\kappa\widetilde{Q}\cdot\widetilde{Q}$ term with $n=2$.

In order to reveal the crossover region more precisely, similar to the analysis[21–22] for the Bose systems, overlaps of the excited states of Hamiltonian (6) for given n and J as functions of the control parameter x with those of the corresponding limiting cases $|\langle nJ_\zeta=4_1;x=x_0|nJ_\zeta=4_1;x\rangle|$ with $x_0=0$ and $x_0=1$ including that corresponding to different total number of particles n in the $j=15/2$ shell are calculated, of which the results for $n=4,6,8$ cases are shown in Fig. 3.

It can be seen from Fig. 3 that when n is small,

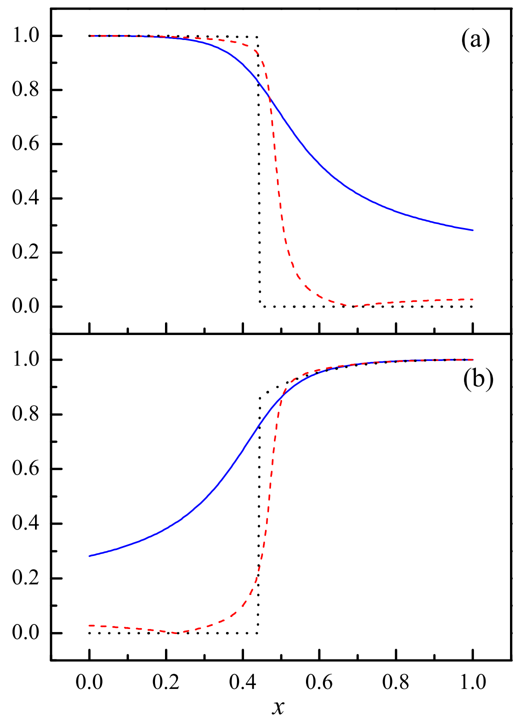

Fig. 3　(color online) The overlaps $|\langle n4_1;x=x_0|n4_1;x\rangle|$ with $x_0=0$ and $x_0=1$ for n=4,6,8 in the $j=15/2$ shell. (a) The overlap $|\langle n4_1;x=0|n4_1;x\rangle|$. (b) The overlap $|\langle n4_1;x=1|n4_1;x\rangle|$. The solid, dash and dot line represent $n=4$, $n=6$ and $n=8$ particles, respectively.

the change of overlap is smooth, and becomes sharper towards the half-filling. The position of the crossover point differ with n but all within the $x \sim 0.4$-0.55 region. A sharp change appear in $|\langle n4_1; x = x_0 | n4_1; x\rangle|$ with $x_0 = 0$ or $x_0 = 1$ around $x_c = 0.44$ when $n = 8$. There is only a crossover region within which the overlaps, energy ratios, and other quantities may change noticeably.

As is well known from the analysis of the quantum phase transitions in Bose systems[21–22], some B(E2) ratios may be sensitive to the shape phase transitions. Several B(E2) values and ratios of the model as functions of x for n =4, 6, and 8 in the j=15/2 shell are presented in Fig. 4, which show that these quantities undergo noticeable change within the crossover region. In our calculation, the E2 transition operator is simply taken as the geometric quadrupole operator defined in Eq. (2). Fig. 4 (a) shows $B(\mathrm{E2};4_1\to2_1)/B(\mathrm{E2};2_1\to0_g)$ gradually increases with the increasing of x when $x < 0.4$ and then becomes decreasing with the increasing of x when $x > 0.4$. On the contrary, in Fig. 4 (b), there is a sudden increasing in the ratio $B(\mathrm{E2};4_2\to2_1)/B(\mathrm{E2};2_1\to0_g)$ within the crossover region. From Fig. 4 one can see that its change is smooth when n is small, and becomes sharper towards the half-filling. With the increasing of the number of particles, there are more B(E2) values and their ratios undergo noticeable change though other B(E2) values and ratios are not provided in Fig. 4. Moreover, the crossover region in the B(E2) ratios becomes rather narrower when the number of particles reaches to the half-filling when n is even. As clearly shown in Fig. 4 for the even n cases, the crossover region becomes a point around $x_c{\sim}0.43$, at which both the B(E2) values and the ratios undergo a sudden change.

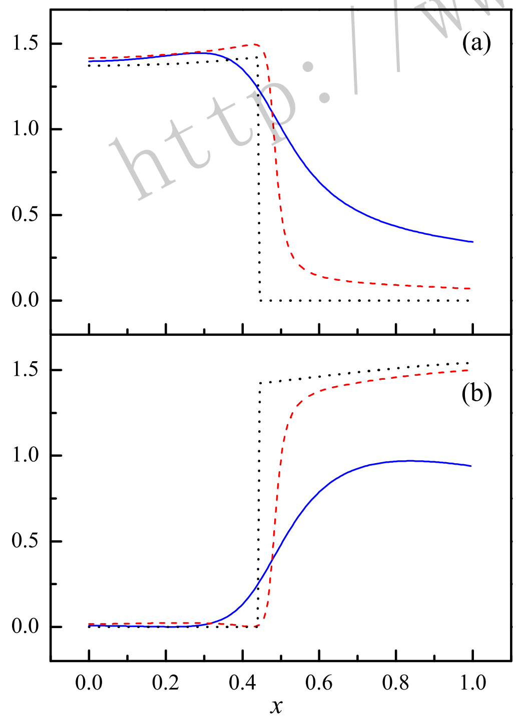

Fig. 4 (color online) Several B(E2) ratios as functions of x for n=4,6, and 8 particles in the j=15/2 shell. (a) $B(\mathrm{E2};4_1\to2_1)/B(\mathrm{E2};2_1\to0_g)$ as functions of x. (b) $B(\mathrm{E2};4_2\to2_1)/B(\mathrm{E2};2_1\to0_g)$ as functions of x. The solid, dash and dot line represent n=4, n=6 and n=8 particles, respectively.

Besides B(E2) ratios, ratios of electric quadrupole moments $Q(J_\zeta)$ may also be used to identify the crossover. Similar to the B(E2) ratios, there are many quadrupole moment ratios of low-lying states may undergo noticeable change within the crossover region though only three quadrupole moment ratios of low-lying states for $n=6$ as functions of x for $j=15/2$ are shown in Fig. 5, which shows that these ratios undergo drastic change around the crossover point.

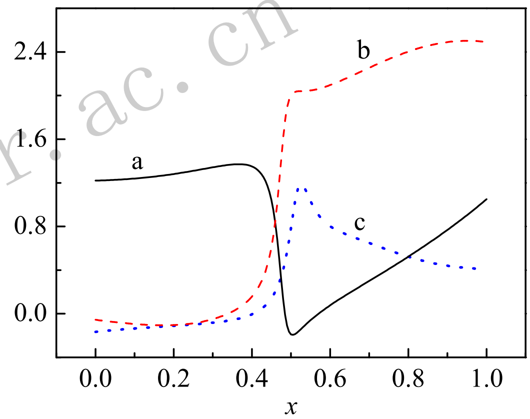

Fig. 5 (color online) Several electric quadrupole moment ratios of low-lying states for $n = 6$ in the $j = 15/2$ shell as functions of x, where the curve a represents $Q(4_1)/Q(2_1)$, the curve b represents $Q(4_2)/Q(2_1)$, and the curve c represents $Q(2_3)/Q(2_1)$. Taken from Ref. [19].

4 Summary

In this paper, the QPC behavior of the spherical shell model mean-field plus the geometric quadrupole-quadrupole and standard pairing model within a single-j shell is analyzed in detail. Our analysis shows that the spectrum generated by the geometric quadrupole-quadrupole interaction does not follow the $J(J+1)$ law, especially in the higher lying states of the yrast band when the number of particles and the value of j is small, which may be due to the fact that the geometric quadrupole operator is less collective in comparison to the Elliott's dynamic counterpart. Only when j is sufficiently large, the low-lying part of the spectrum generated by the geometric quadrupole-quadrupole interaction follow the $J(J+1)$ law even when there are only a few particles. Due to the Pauli exclusion, only a crossover occurs in the model because j is small and finite. Various quantities, such as low-lying excitation spectrum, the overlaps of the excited states with those of the corresponding limiting cases, ratios of $B(\mathrm{E2})$ values and electric quadrupole moments of some low-lying states, as functions of the control parameter of the model in the j=15/2 shell are calculated. The results show that there are noticeable changes within the crossover region of the rotation-like to the pair-excitation phase transition, especially in the half-filling case. The crossover region becomes narrower with the increasing of the number of particles toward the half-filling, especially in the even j cases.

A chain of isotopes Sn with valence neutrons confined in the $0g_{7/2}$ shell less than 8 may show the properties discussed in the present model of single-j shell. In order to investigate the quantum phase crossover of neutron-rich Sn isotopes, a multi-j extension of the current model is necessary, in which other seniority conserving interaction terms[14, 23] may also be considered. These extensions will be a part of our future work.

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球形平均场加四极-四极和对力模型在单 j 壳内的量子相交叉行为

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摘要： 介绍了单j壳的球形平均场加几何四极-四极和标准对力模型的量子相交叉行为。在单$j = 15/2$的壳内，计算了随模型控制参数变化的多个物理量如低激发能级、激发态间重叠积分、低激发态间的B(E2)比值和电四极矩比值。结果显示，在类转动到对激发相的演化中，多个物理量在交叉区存在非常明显的变化，如$B(\mathrm{E2};4_1 \to 2_1)/B(\mathrm{E2};2_1 \to 0_g)$，$B(\mathrm{E2};4_2 \to 2_1)/B(\mathrm{E2};2_1 \to 0_g)$等，并且这些变化在核子数达到半满壳时尤为显著。此外，尽管当j较小时，由几何四极-四极相互作用得到的低激发能级不满足转动谱规律，但当j值足够大时，这些低激发能级满足转动谱规律。

关键词： 壳模型；四极-四极相互作用；单j壳

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