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Quantum Phase Transition in an Extension of the Interacting Boson Model Based on Dual Algebraic Structure

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Abstract: An extension of the original interacting boson model to the multi-level case including negative parity f- and p-bosons is made. An affine $\widehat{SU(1,1)}$ algebraic approach is applied to solve the multi-level pairing problem numerically via the dual algebraic structure. The duality relation is explicitly used to construct the number-conserving unitary and number-nonconserving quasi-spin algebra, related with the Hamiltonian and the corresponding bases. After fitting to the experimental level energies of even-even $^{106-116}$Cd, several order parameters to signify the shape (phase) transition, such as occupation numbers of the bosons in the ground and a few lowest excited states, the level energy staggering in the (quasi)-γ band, are calculated to demonstrate the shape (phase) transitional behavior of these medium mass transitional nuclei.

Key words: negative parity states; affine $\widehat{SU(1,1)}$ algebra; dual algebraic structure; signature of the phase transition.

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1 Introduction

Quantum phase transitions (QPT) in different systems are known to have many characteristics in common. Numerous studies of QPT in nuclear models have been reported in recent years[1–5]. For several years, great attention has been devoted to the study of the nuclear structures as well as QPT in low-lying ground and excited states of nuclei within the framework of the two-level[6–11], three-level[12–15], and four-level pairing[16–24] model in the interacting boson model (IBM) framework. The role played by d-bosons is easily understood since they can be thought of as a quantization of the shape variables. The introduction of s-bosons is less obviously necessary and arose from a study of the underlying microscopic structure which led to an interpretation of bosons in terms of nucleon pairs. Although d- and s-bosons dominate the low-lying features of nuclei, other degrees of freedom, such as f- and p-bosons are also necessary in order to describe negative parity excited states.

In this talk, we show how the octupole and dipole degrees of freedom are introduced in the IBM for nuclei in the vibrational to γ-unstable transitional region. We report our recent applications of the infinite dimensional affine $\widehat{SU(1,1)}$ algebraicc approach[25–26] to solve the pairing interactions among s-, d-, p-, and f-bosons as extensions of the IBM. The nuclear shape among which phase transitions take place is associated with the vibrational and γ-soft dynamical symmetries of the IBM. When p- and f-bosons are introduced alongside s- and d-bosons, the space spanned by single boson states becomes 16-dimensional. The corresponding dynamical group is $U(16)$[27]. We use the $U_{\rm f}(7)\otimes U_{\rm d}(5)\otimes U_{\rm p}(3)\rightarrow SO(16)$ transitional description and introduce the related affine $\widehat{SU(1,1)}$ algebraic technique, with which we calculate level energies of even-even $^{106-116}$Cd and show signatures of the shape (phase) transition in these nuclei.

2 The method

Although quite successful in its simplest form of the original IBM, many aspects of nuclear structure still could not be described with scalar and quadrupole

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bosons only. To describe these aspects, several extensions have been made. The quadrupole vibration and deformation are described in terms of s- and d-bosons (with angular momentum l=0 and l=2), while negative parity states may be described by introducing f- and p-bosons (l=3 and l=1). In the IBM framework, the boson quasi-spin algebra was first introduced in Ref. [28]. Accordingly, the scalar pair creation operator $S^+(l)$, pair annihilation operator $S^-(l)$, and number-conserving operator $S^0(l)$ generate the $SU^l(1,1)$ algebra, which is dual to the $Sp(4l+2,R) \supset U(2l+1) \supset SO(2l+1) \supset SO(3)$ for the bosonic systems.

It is known that pairing plays a prominent role in understanding the nuclear many-body systems[29–32]. The concept of pairing interaction was first reported by Racah within the context of a seniority coupling scheme[33]. The bosonic pairing model can be solved exactly by the Bethe ansatz method which usually depends on a set of dimensionless parameters. The Bethe ansatz method is the process of diagonalizing a many-body Hamiltonian starting from a suitably constructed ansatz state and eventually determining the exact eigenstates in an explicit way[25–26].

In order to analyze the structural changes between the spherical and γ-unstable shapes via Bethe ansatz method, similar to the procedure shown in Refs. [25–26], we introduce the $SU^l(1,1)$ generators denoted as

$$S^+(l)=\frac{1}{2}l^\dagger\cdot l^\dagger,\quad S^-(l)=\frac{1}{2}\tilde{l}\cdot\tilde{l},$$
$$S^0(l)=\frac{1}{2}(l^\dagger\cdot\tilde{l}+\frac{2l+1}{2})=\frac{1}{2}(n_l+\frac{2l+1}{2}),\qquad(1)$$

where $l^\dagger$ is the creation operator of l-bosons, $\tilde{l}_\nu=(-1)^\nu l_{-\nu}$, and n_l is the l-boson number operator, which are the pair creation operator $S^+(l)$, pair annihilation operator $S^-(l)$, and number-conserving operator $S^0(l)$, with l=0 for s-boson, l=1 for p-boson, $l=2$ for d-boson, and l=3 for f-boson. The structure of the dual algebraic relations including the generators of the related unitary and quasi-spin algebras and the Casimir operators are presented in Ref. [10]. In the dual reductive pairs, duality relationships are often described in terms of a pair of two algebras (groups) that are mutually commutative with each other.

Similar to Refs. [25–26], we can introduce the affine $\widehat{SU^{spdf}(1,1)}$ algebra generated by

$$S_n^\pm=\sum_l c_l^{2n+1}S^\pm(l),\quad S_n^0=\sum_l c_l^{2n}S^0(l),$$

with $n=0,\pm1,\pm2,\cdots$, where $c_0=c_{\rm s}$, $c_1=c_{\rm p}$, $c_2=c_{\rm d}$, and $c_3=c_{\rm f}$ are real parameters. Then, the schematic Hamiltonian suitable to describe the vibrational to γ-unstable shape (phase) transition with s-, p-, d-, and f-boson degrees of freedom is expressed as

$$\hat{H}=gS_0^+S_0^-+\alpha S_1^0+\beta\hat{C}_2(SO_{\rm f}(7))+\gamma\hat{C}_2(SO_{\rm d}(5))+\delta(\hat{C}_2(SO_{\rm pdf}(3))),\qquad(2)$$

of which the first two terms $S_0^+S_0^-$ and S_1^0 are related to the $SU(1,1)$ pairing interactions among the bosons, while other Casimir operators are already diagonal under the $SO_{\rm f}(7)\otimes SO_{\rm d}(5)\otimes SO_{\rm p}(3)\supset SO_{\rm pdf}(3)$ basis. The dynamical group chain related to the Hamiltonian (2) is

$$U_{spdf}(16)\supset\left\{\begin{array}{c}SO_{spdf}(16)\\ U_{\rm f}(7)\otimes U_{\rm d}(5)\otimes U_{\rm p}(3)\otimes U_{\rm s}(1)\end{array}\right\}\supset SO_{\rm f}(7)\otimes SO_{\rm d}(5)\otimes SO_{\rm p}(3)\supset SO_{\rm f}(3)\otimes SO_{\rm d}(3)\otimes SO_{\rm p}(3)\supset SO_{pdf}(3).\qquad(3)$$

It can be easily seen that Eq. (2) is diagonal under the $U_{spdf}(16)\supset U_{\rm f}(7)\otimes U_{\rm d}(5)\otimes U_{\rm p}(3)\otimes U_{\rm s}(1)\supset SO_{\rm f}(7)\otimes SO_{\rm d}(5)\otimes SO_{\rm p}(3)\supset SO_{pdf}(3)$ basis when $g=0$, while it is diagonal under the $U_{spdf}(16)\supset SO_{spdf}(16)\supset SO_{\rm f}(7)\otimes SO_{\rm d}(5)\otimes SO_{\rm p}(3)\supset SO_{pdf}(3)$ basis when $c_{\rm s}=c_{\rm d}=c_{\rm p}=c_{\rm f}$ for arbitrary value of g. In the following, $c_{\rm d}=1$ is fixed, while $c_{\rm s}$, $c_{\rm p}$, and $c_{\rm f}$ vary in the closed interval $[0,1]$. Therefore, (2) describes nuclei within the $U_{\rm f}(7)\otimes U_{\rm d}(5)\otimes U_{\rm p}(3)\longleftrightarrow O_{spdf}(16)$ vibration to γ-unstable transitional region with inclusion of dipole and octupole degrees of freedom.

According to the Bethe ansatz method, up to a normalization constant, the eigenstates of (2) may be expressed in terms of the $U_{spdf}(16)\supset U_{\rm f}(7)\otimes U_{\rm d}(5)\otimes U_{\rm p}(3)\otimes U_{\rm s}(1)\supset SO_{\rm f}(7)\otimes SO_{\rm d}(5)\otimes SO_{\rm p}(3)\supset SO_{pdf}(3)$ vectors with the highest weight of $SO_{\rm f}(7)\otimes SO_{\rm d}(5)\otimes SO_{\rm p}(3)$ as

$$|k;\nu_{\rm s}\nu_{\rm p}\nu_{\rm d}\nu_{\rm f}n_\Delta LM\rangle=\prod_{\rho=1}^{k}S^+(x_\rho)|\nu_{\rm s}\nu_{\rm p}\nu_{\rm d}\nu_{\rm f}n_\Delta LM\rangle,\qquad(4)$$

where $\nu_{\rm s}$, $\nu_{\rm p}$, $\nu_{\rm d}$, and $\nu_{\rm f}$ are seniority number of s-, p-, d-, and f-bosons, respectively, n_Δ is a set of additional quantum numbers needed in the reduction, and L is the total angular momentum quantum number of the *spdf*-boson system, the total number of bosons is given by $N=2k+\sum_l\nu_l$, and

$$S^+(x)=\sum_l\frac{c_l}{1-c_l^2x}S^+(l),\qquad(5)$$

in which x is the spectral parameter. The eigenenergies and the related k-variables $\{x_1,\ \cdots,\ c_k\}$ in Eq. (4) are determined by the corresponding Bethe ansatz equations as shown, for example, in Refs. [6, 31].

3 Theoretical results

The medium mass nuclei are typical examples of transitional class C nuclei. Nuclei in this region have transitional characteristics intermediate between spherical and γ-unstable shapes[12, 25, 32]. Very recently, these regions were also studied in the generalized collective[34] and Hartree-Fock-Bogoliubov[35–37] models. In Ref. [38] the even-even Cd isotopes were considered within the framework of the two-, three-, and four-level bosonic pairing models with the aim of identifying states having a quadrupole-octupole characters.

As examples of the present model application, level energies and some effective order parameters in signifying the QPT in $^{106-116}$Cd were calculated in the present model. In our calculation, we take $c_{\rm d}=1$, while $c_{\rm s}=c_{\rm p}=c_{\rm f}=c$, which varies within in closed interval $c\in[0,1]$. The parameters of the Hamiltonian (2) are obtained from the least square fit to the experimentally determined level energies of these nuclei. Also, since the IBM is designed for low-lying states with energies almost less than 3.5 MeV, level energies higher than 3.5 MeV are not considered in our fits. The calculated level energies from the model along with the experimental ones are shown in Fig. 1. In order to reduce parameters, we mainly concerned with the overall fitting quality with as few parameters as possible. For example, the parameters g is fixed for all nuclei fitted, and other parameters are expressed as a function of total number of bosons, which are explicitly shown in the caption of Fig. 1. Once the model parameters are determined by the energy spectra as shown in the caption of Fig. 1, electromagnetic transition rates, such as B(E2), B(E3), and B(E1), may also be calculated, for example, as done so in the same model for Pd isotopes shown in Ref. [6]. The fitting quality of B(E2) values of even-even $^{106-116}$Cd is similar to that of the sd-IBM shown previously[25].

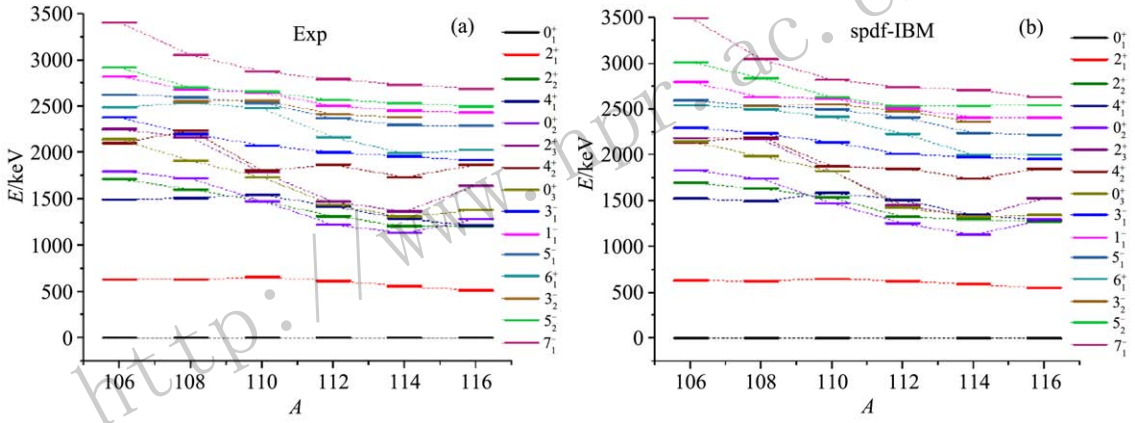

Fig. 1 (color online) Level energies of even-even $^{106-116}$Cd, where the experimental data (a) are taken from Refs. [39–44], while the parameters used in the spdf-IBM fits (b) are $g=1$ keV, $\alpha=1000$ keV, $\beta=24{\rm e}^{-0.8N}$ keV, $\gamma=53{\rm e}^{-0.51N}$ keV, and $\delta=78{\rm e}^{-0.14N}$ keV, in which N is the total number of bosons of the corresponding nucleus, the parameter $c=0.43$, 0.42, 0.55, 0.52, 0.50, and 0.38, respectively, were used for even-even $^{106-116}$Cd in the fitting.

It is well known that the ground$d$-state occupation number of d-bosons is an effective order parameter to signify the shape (phase) transitions in the IBM. The occupation number of d-, p-, and f-boson for a given state ψ defined by

$$\langle\hat{n}_l\rangle=\frac{\langle\psi|\hat{n}_l|\psi\rangle}{N} \tag{6}$$

as functions of the control parameter $c_{\rm s}$ and $c_{\rm p}$ may be calculated to show the phase transitional behavior, where $|\psi\rangle$ is the ground state or excited state of the model determined according to the eigenstate shown in EQ. (4).

As examples, $\langle\hat{n}_{\rm d}\rangle$ and $\langle\hat{n}_{\rm p}\rangle$ for the ground state as functions of the control parameters $c_{\rm s}$ and $c_{\rm p}$ are shown in Fig. 2, for which the other parameters were chosen according to the fits to the level energies of ^{114}Cd shown in Fig. 1, while $c_{\rm d}=1$ and $c_{\rm f}=0.9$ are taken in the plots. As clearly shown in Fig. 2, that the control parameter $c_{\rm s}$ affects the d-boson occupation number significantly, while $\langle\hat{n}_{\rm d}\rangle$ is almost unaffected by the control parameter $c_{\rm p}$. The obvious sudden decease in $\langle\hat{n}_{\rm d}\rangle$ with the increasing of $c_{\rm s}$ can be observed around $c_{\rm s}\sim0.5$. Moreover, both $c_{\rm s}$ and $c_{\rm p}$ affect the p-boson occupation number as shown in the lower panel of Fig. 2. It can be inferred from Fig. 2 that both the d-boson and p-boson occupation number seem more sensitive to the control parameter $c_{\rm s}$.

In addition, the variations of energy spectra may also be taken as a signature of the shape (phase). In

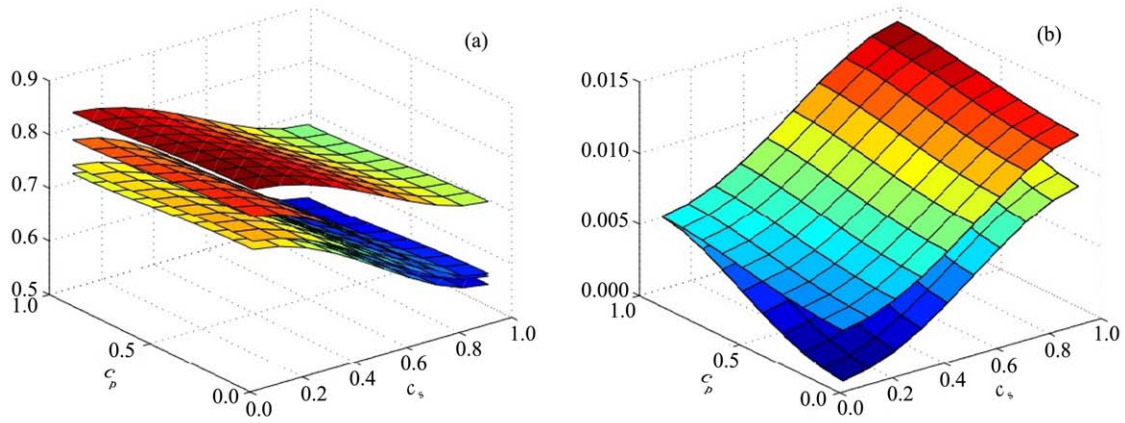

Fig. 2　(color online) The d-boson (a) and p-boson (b) occupation numbers of a few low-lying states as functions of the control parameters $c_{\rm p}$ and $c_{\rm s}$.

the past a few years, there has been a growing interest in the staggering pattern as a key signature of the γ-softness or triaxial shape (phase)[45–46]. In the present work, we employ the three-term nearest neighbor level-energy difference within the (quasi)-γ band defined by

$$S(L)=\frac{[E(L)_\gamma-E(L-1)_\gamma]-[E(L-1)_\gamma-E(L-2)_\gamma]}{E(2_1^+)}, \tag{7}$$

where γ is referred to the (quasi)-γ band. The staggering in $S(L)$ with L is a main character of the γ-softness, which occurs, for example, in the critical point nuclei ^{112,114}Cd. Fig. 3 shows that the staggering pattern in ^{114}Cd can indeed be reproduced by the $sdpf$-IBM with both $c_{\rm p}\neq 0$ and $c_{\rm p}=0$, of which the results are denoted as $spdf$ and sdf, respectively.

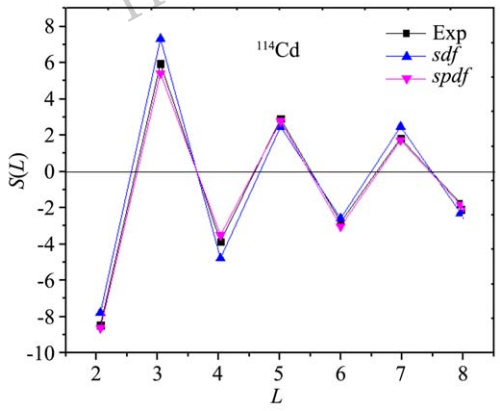

Fig. 3　(color online) Staggering pattern $S(L)$ for ^{114}Cd, where the experimental data are taken from Ref. [43].

4　Summary and conclusion

In this contribution, exact solutions of the four-level pairing model with inclusion of f- and p-bosons are used to investigate the shape (phase) transitional behavior of nuclei within the spherical to γ-unstable transitional region. Low-lying level energies with both positive and negative parity, the occupation number of bosons, and nearest neighbor level-energy difference within the (quasi)-γ band in even-even $^{106-116}$Cd were calculated. It can be observed that the fitting quality of this model to the lower part of the energy spectra of these nuclei are quite good, and the staggering pattern in the (quasi)-γ band of Cd is well reproduced. As far as the occupation numbers of the bosons are concerned, it seems that both the d- and p-boson occupation numbers can be taken as effective order parameters to signify the shape (phase) transition within the vibrational to γ-unstable transitional region. It should be pointed out that other quantities, such as energy ratios, electric quadrupole, octupole, and magnetic dipole transitions, *etc.* should also be analyzed in order to get a better understanding of the (bosonic) pairing effect to the shape (phase) transition based on the dual algebraic structure of the model.

References:

[1] BUCURESCU D, ZAMFIR N V. Phys Rev C, 2018, **98**: 024301.
[2] PAN F, LI D, CHENG G, *et al.* Phys Rev C, 2018, **97**: 034316.
[3] PAN F, YUAN S, QIAO Z, *et al.* Phys Rev C, 2018, **97**: 034326.
[4] ZHANG Y, PAN F, LIU Y X, *et al.* Phys Rev C, 2017, **96**: 034323.
[5] PAVEL C, PAVEL S. Phys Scr, 2016, **91**: 083006.
[6] MARJARSHIN A J, JAFARIZADEH M A, SABRI H. *et al.* Eur Phys J Plus, 2017, **132**: 418.
[7] GREINER W , MARUHN J A. Nuclear Models[M]. Berlin: Springer, 1996.
[8] IACHELLO F AND ARIMA A. The Interacting Boson Model[M].Cambridge: Cambridge University Press, 1987.
[9] HEYDE K. in Algebraic Approaches to Nuclear Structure [M]. Swizerland: Hardwood Academic Publishers, 1993.

[10] CAPRIO M, SKRABACZ J, IACHELLO F. J Phys A: Math Theor, 2011, **44**: 075303.
[11] CEJNAR P, STRANSKY P, KLOC M. Phys Scr, 2015, **90**: 114015.
[12] JAFARIZADEH M A, MARJARSHIN A J, FOULADI N. Int J Mod Phys E, 2016, **25**: 1650089.
[13] VON BRENTANO P, ZAMFIR N, ZILGES A. Phys Lett B, 1992, **278**: 221.
[14] JUNGCLAUS A, BORNER H G, JOLIE J, *et al.* Phys Rev C, 1983, **47**: 1020.
[15] S. LERMA H., ERREA B, DUKELSKY J, *et al.* Phys Rev C, 2006, **74**: 024314.
[16] SPIEKER M, BUCURESCU D, ENDRES J, *et al.* Phys Rev C, 2013, **88**: 041303.
[17] MARJARSHIN A J, JAFARIZADEH M A. Nucl Phys A, 2017, **968**: 287.
[18] ZAMFIR N V, KUSNEZOV D. Phys Rev C, 2001, **63**: 054306.
[19] KUYUCAK S, HONMA M. Phys Rev C, 2002, **65**: 064323.
[20] LONG G L, SHEN T Y, JI H Y, *et al.* Phys Rev C, 1998, **57**: 2301.
[21] JAFARIZADEH M A, MARJARSHIN A J, FOULADI N, *et al.* J Phys G: Nucl Part Phys, 2016, **43**: 095108.
[22] SPIEKER M, PASCU S, ZILGES A, *et al.* Phys Rev Lett, 2015, **114**: 192504.
[23] GARRETT P, LEHMANN H, JOLIE J, *et al.* Phys Rev C, 1999, **59**: 2455.
[24] PASCU S, ENDRES J, ZAMFIR N V, *et al.* Phys Rev C, 2012, **85**: 064315.
[25] PAN F, DRAAYER J P. Nucl Phys A, 1998, **636**: 156.
[26] PAN F, ZHANG X, DRAAYER J P, J Phys A: Math Gen, 2002, **35**: 7173.
[27] KUSNEZOV D. J Phys A: Math Gen, 1990, **23**: 5673.
[28] UI H. Ann Phys, 1968, **49**: 69.
[29] DAI L, PAN F, DRAAYER J P. Nucl Phys A, 2017, **957**: 51.
[30] PAN F, ZHOU D, DAI L R, *et al.* Phys Rev C, 2017, **95**: 034308.
[31] MARJARSHIN A J, SABRI H. Nucl Phys A, 2017, **964**: 69.
[32] MARJARSHIN A J. Eur Phys J A, 2018, **54**: 11.
[33] RACAH G. Phys Rev, 1942, **62**: 438.
[34] TROLTENIER D, MARUHNW J A, GREINER W, *et al.* Z Phys A, 1991, **338**: 261.
[35] BHARTI A, DEVI R, KHOSA S K. J Phys G: Nucl Part Phys, 1994, **20**: 1231.
[36] SINGH A J, RAINA E K. Phys Rev C, 1995, **53**: 1258.
[37] BHARTI A, KHOSA S K. Phys Rev C, 1996, **53**: 2528.
[38] GIANNATIEMPO A, NANNINI A, SONA P, *et al.* Phys Rev C, 1995, **52**: 2969.
[39] DE FRENNE D, NEGRET A. Nucl Data Sheets, 2008, **109**: 943.
[40] BLACHOT J. Nucl Data Sheets, 2000, **91**: 135.
[41] DE FRENNE D, JACOBS E. Nucl Data Sheets, 2000, **89**: 481.
[42] DE FRENNE D, JACOBS E. Nucl Data Sheets, 1996, **79**: 639.
[43] BLACHOT J. Nucl Data Sheets, 2012, **113**: 515.
[44] BLACHOT J. Nucl Data Sheets, 2010, **111**: 717.
[45] MCCUCHAN E, BONATSOS D, ZAMFIR N V, *et al.* Phys Rev C, 2007, **76**: 024306.
[46] CHABAB M, LAHBAS A, OULNE M. Eur Phys J A, 2015, **51**: 1.

推广的相互作用玻色子模型中基于对偶代数结构的量子相变研究

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摘要： 本工作将相互作用玻色子模型推广为包含 f- 和 p- 玻色子的情形。利用仿射型$\widehat{SU(1,1)}$代数方法，通过对偶代数结构数值计算了多分量玻色型对力问题。利用对偶关系解析构建了与哈密顿量及其基底相联系的，由幺正的粒子数守恒和非粒子数守恒算符构成的准旋代数。在经该模型对$^{106-116}$Cd 偶偶核素实验能谱拟合的基础上，计算了基态和低激发态中各种玻色子占有率，准 γ 带中相邻能级摇摆等几个能特征该区域核素形状相变的序参量。从而展示了这些中重质量核从振动到 γ- 不稳定运动的形状相变行为。

关键词： 负宇称态；仿射型 $\widehat{SU(1,1)}$ 代数；对偶代数结构；相变特征量

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