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Correlations of the $SO(8)-$Pairing and $SU(3)-$Quadrupole Bases in the Algebraic Shell Model

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Abstract: We establish a correspondence between the $SO(8)$ isoscalar, isovector and total pairing bases and the Elliott's $SU(3)$ basis in the algebraic structure of the spatial part of the microscopic shell model. It is derived from the complementarity of these algebras to the same T, S, (S,T) irreducible representations (irreps) of the Wigners supermultiplets, contained in the shell-model number-conserving algebra $U(4\Omega)$. This important result allows for the evaluation of the content of $SU(3)$ irreps into the different types of pairing bases which leads to an investigation of the complementarity and competitive effects of pairing and the quadrupole-quadrupole interactions on the energy spectra of the nuclear systems. The theory is valid for any shell and for a number of shells as well, but we illustrate it with the results for a single $ds-$shell.

Key words: algebraic model; pairing interaction; quadrupole-quadrupole interaction; shell model; spin-isospin space

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1 Introduction

It has been understood since the early years of the development of the nuclear structure physics, that the pairing[1] and the quadrupole-quadrupole interactions[2] are the most important short- and long-range interactions that have to be taken into account in the shell-model description of the nuclear systems[3]. Being with different range of action on the nucleons in the valence shells it is quite clear that these interactions actually influence the behavior of the systems in different parts of the shells. The pairing interaction is responsible for the appearance of the pairing gap in the nuclei with only a few nucleons after the closed shells and is therefore associated with the spherical shape of the system. The quadrupole-quadrupole interaction dominates in the nuclei near the mid-shells and so introduces deformation, which is related to the appearance of rotational sequences in the nuclear spectra. Hence, in some nuclei each of these interactions could reproduce relatively well the observed behavior of the nuclear system, but in most of the cases the study of the relationship between them is of great importance. This is the main motivation for the development of the Pairing-plus-Quadrupole Model (PQM)[4–6] for the description of the nuclear excitation spectra. It is most successfully done in the framework of the basic shell model representation of the employed interactions, but in this case the applications to real nuclear systems are rather complicated and cumbersome, due to the enormous dimensionality of the basis space in particular for the heavy nuclei. It is already clearly proven that such a problem is easily avoidable by employing a group-theoretical approach[7], which introduces symmetry principles useful in particular for reducing the basis spaces and in the calculation of matrix elements of transitional operators.

In this work we present a symmetry approach by introducing the pairing and quadrupole-quadrupole interactions as invariants of respective algebras, which reduce the space symmetry of the shell model in a dynamical way to the $SO(3)$ algebra of the angular momentum L. The so defined dynamical symmetry chains are simultaneously complementary to the Wigner's spin-isospin $SU_{ST}(4)$, the spin $SU_S(2)$ or isospin $SU_T(2)$ symmetry, which establishes the direct connection between these two limiting cases. Since the exactly solvable limiting cases defined by the so ob-

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tained dynamical symmetries can be considered as different phases of the nuclear collectivity, this approach allows for the investigation of the competing and complementarity features[8] of the pairing and quadrupole interactions in the description of the realistic nuclear systems in the lower shells up to mass numbers $A \sim 100$. We illustrate the applications of this extended algebraic realization[9] of the PQM theory for nuclei with four valence particles in the *ds* shell. These examples do not represent the general case of the shell model spaces, but in it we are able to not only reproduce the experimental data, but as well to evaluate the weight of each of the considered modes on the nuclear structure. The two modes - the pairing and the quadrupole interactions - compete, and in this way different types of collective spectra, ranging between vibrational and rotational can be investigated in terms of phase transitions[8].

2 The algebraic microscopic shell-model

The many-particle shell model wave functions are constructed by filling the single-particle orbitals of the valence shells with nucleons, taking into account the Pauli principle. The latter constrain is imposed by requiring an antisymmetrization of the total wave function, containing the product of the spatial, spin and isospin parts. In general, this condition and the complementarity of the particle permutation symmetry group and the unitary transformation on the state orbitals[7], allows the use of the simpler case of $U(4\Omega)$ for the classification of the $m-$ particles' wave functions. In $U(4\Omega)$ the number 4 stands for the dimensionality of the spin-isospin space and $\Omega = \sum_i (2l_i + 1)$ is the dimensionality of the considered shell-model valence space, generated by the $\boldsymbol{LS}$-coupling of m nucleons in the $l_1, l_2 \ldots l_r$ orbits of the considered shells.

As a result, the antisymmetric irreducible representations of $U(4\Omega)$ for m particles, labeled by the Young diagrams $\{1^m\}$ can be further partitioned into spin-isospin and spatial parts

$$\begin{array}{lll} U(4\Omega) \supset & U_{ST}(4) \otimes & U(\Omega) \\ \{1^m\} & \{\tilde{f}\} & \{f\}\ , \end{array} \qquad (1)$$

under the condition that each of the possible irreps $\{f\} \equiv \{f_1, f_2, f_3, f_4\}$ $(f_1 \geqslant f_2 \geqslant f_3 \geqslant f_4)$ and $\{\tilde{f}\} \equiv \{\tilde{f}_1, \tilde{f}_2, \tilde{f}_3, \tilde{f}_4\}$, where $\tilde{f}_1 \geqslant \tilde{f}_2 \geqslant \tilde{f}_3 \geqslant \tilde{f}_4$ of the two complementary groups $U_{ST}(4)$ and $U(\Omega)$ respectively are conjugated to each other by interchange of the rows and columns in their respective Young tableaux. Consequently, both representations can be obtained from each other. Since they are contained in the simple representation $\{1^m\}$ with $m = \sum_i \tilde{f}_i$, shell-model wave functions are only labeled with the number of particles m in the valence shell.

2.1 The spin and isospin symmetries in the shell model

The way to account for the spin and isospin symmetries in the niclear structure was suggested by Wigner[10], and is based on the invariance of the nuclear forces in respect to rotations in spin and isospin spaces. This invariance is introduced through the Lie algebra of $SU_{ST}(4)$, whose representations $\{\tilde{f}\} \equiv \{\tilde{f}_1 - \tilde{f}_2, \tilde{f}_2 - \tilde{f}_3, \tilde{f}_3 - \tilde{f}_4\}$ are conjugated to and can be obtained by the representations $\{f\}$ of $U(\Omega)$. It is obvious that the energy of the nuclear states strongly depends on these quantum numbers.

Wigner's supermultiplet model is actually the nuclear $\boldsymbol{LS}$-coupling scheme[7, 11], which employs the reduction

$$\begin{array}{lll} SU_{ST}(4) \supset & SU_S(2) \otimes & SU_T(2) \\ \{\tilde{f}\} & S & T\ , \end{array} \qquad (2)$$

that gives the total spin $S = \sum_i s_i$ and isospin $T = \sum_i t_i$ (i enumerates the considered particles) values of the states wave functions, which together with the orbital angular momentum $L = \sum_i l_i$ of $SO_L(3)$ are good quantum numbers in this case. Further, L and S could be coupled to total angular momentum $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$. Using the isomorphism of the algebras $SU_{ST}(4) \sim SO_{ST}(6)$, $SU_S(2) \sim SO_S(3)$ and $SU_T(2) \sim SO_T(3)$, another shell-model reduction chain, equivalent to the chain (2) can be identified[12]:

$$\begin{array}{llll} SU_{ST}(6) & \supset & SO_S(3) \otimes & SO_T(3) \\ [P_1, P_2, P_3] & & S & T\ , \end{array} \qquad (3)$$

and used in a conjunction with the spatial reduction of $U(4\Omega)$. We should point out, that the $SU_{ST}(4)$ symmetry is broken in a non-dynamical way by the Coulomb and the $l.s$ interaction[11] in the nuclear mean-field approach. The role of both of these increases with the nuclear mass number, but the considered $\boldsymbol{LS}$-coupling scheme is still applicable to nuclei up to mass $A \approx 100$. Within the number-conserving shell-model algebra $U(4\Omega)$, we consider two other reduction chains, which are related to the pure isoscalar and isovector pairing interactions. In these cases, one starts with the reduction:

$$\begin{array}{lll} U(4\Omega) \supset & U_\sigma(2) \otimes & U(2\Omega) \\ [1^m] & \{\tilde{f}_\sigma\} & \{f_\sigma\}\ , \end{array} \qquad (4)$$

where $\sigma = S$ or T. In the above reduction the labels of the representations given below the subalgebra, are obtained by the standard reduction rules:$\{\tilde{f}_\sigma\} = \{f_1, f_2\}$, where $f_1 \geqslant f_2 \geqslant 0$, $f_1 + f_2 = m$ and $\sigma = (f_1 - f_2)/2$.

Consequently, the $U(2\Omega)$ irreps $\{f_\sigma\} = \{2^{f_2}, 1^{f_1-f_2}\}$. The necessity of introducing these reductions will be clarified in the next section, where the reductions of the spatial part of $U(4\Omega)$ are presented.

While applying the above reductions the spin S and the isospin T of the system are specified, there is no general rule for obtaining the values of the orbital angular momentum L, contained in the irreps $\{f\}$, since the reduction $U(\Omega) \supset SO_L(3)$ is not a canonical one.

2.2 Reductions of the spatial part

2.2.1 Rotations and the $SU(3)$ symmetry

First, we make use of one of the most important aftermath of the supermultiplet model[13], which can be extended to include in the spherical shell model rotational motion, which is achieved trough the Elliott's $SU(3)$ model[2]. The latter provides an elegant and analytically solvable way for obtaining the missing labels in the reduction of the spatial part of the Wigner's $SU_{ST}(4)$ shell model classification to the orbital angular momentum L, by introducing the reduction[14]:

$$\begin{array}{llll} U(\Omega) \supset & SU(3) \supset & SO(3) & \\ \{f\} \quad \alpha & (\lambda,\mu) & K \quad L & , \end{array} \tag{5}$$

where α indicates the multiplicity of the $SU(3)$ representation (λ,μ) in the $U(\Omega)$ representation $\{f\}$. The $SU(3)$ in Eq. (5) is generated by the components of the quadrupole operator:

$$Q_\mu = \sum_l \sqrt{8(2l+1)} \left(a^\dagger{}_{l\frac{1}{2}\frac{1}{2}} \times \tilde{a}_{l\frac{1}{2}\frac{1}{2}} \right)^{(200)}_{(\mu 00)} \tag{6}$$

and the orbital momentum operator

$$L_\mu = \sum_l \sqrt{4l(2l+1)(l+1)/3} (a^\dagger{}_{l\frac{1}{2}\frac{1}{2}} \times \tilde{a}_{l\frac{1}{2}\frac{1}{2}})^{(100)}_{(\mu 00)} \ , \tag{7}$$

which are presented in a second quantized form. They act in a single valence shell labeled by l and the bracket denotes coupling in the angular momentum, spin and isospin (LST). In addition, the model assumes that the long range residual interaction has a quadrupole character and the quadrupole-quadrupole interaction is written as:

$$V_{\rm rot} = \frac{1}{2} \chi Q \cdot Q \ , \tag{8}$$

where $Q \cdot Q = 4C^2_{SU(3)} - 3L^2$ and the eigenvalue of the second invariant of $SU(3)$ is $C^2_{SU(3)} = \lambda^2 + \lambda\mu + \mu^2 + 3(\lambda + \mu)$. Obviously, Eq. (8) gives rise to a rotational spectra of the type $L(L+1)$. The chain (5) is a classical example of dynamical symmetry breaking of the degeneracy within the $U(\Omega)$ or $SU_{ST}(4)$ by the quadrupole interaction. In this way the rotational states are labeled by the quantum numbers of the representations of the algebras in the chain (5):

$$|\Psi_R\rangle \equiv |\{f\}, \alpha(\lambda,\mu) KLM\rangle \equiv |m, \alpha(\lambda,\mu) KLM\rangle \ , \tag{9}$$

and are obtained in the context of the shell model[11]. Since Elliott's $SU(3)$ model starts with the Wigner's supermultiplet classification, it also breaks down from the spin-orbit term in the nuclear mean field, and causes a considerable rearrangement of the single-particle levels. The model could be applied successfully mainly for nuclei in the ds and fp shells. For treating heavier systems several more refined approaches[11], like the pseudo-spin symmetry[7], have been employed.

2.2.2 Total, isoscalar and isovector pairing

Another way to reduce the multiplicity of the spatial shell-model algebra $U(\Omega)$ to the angular momentum algebra $SO_L(3)$ in the $\boldsymbol{LS}$-coupling scheme is to use the reduction chain:

$$\begin{array}{llll} U(\Omega) \supset & SO(\Omega) \ldots \supset & SO_L(3) & \\ \{f\} & [\tilde{\mu}] & \beta \ L & , \end{array} \tag{10}$$

which can easily be realized using the basic assumption that the fundamental representation $\{1\}_{U(\Omega)}$ is composed by the representations $(l_1, l_2, \ldots l_r)$ of $SO_L(3)$ for nucleons occupying the orbits $l_1, l_2, \ldots l_r$. Then, by using the standard methods for the decompositions $U(n) \supset O(n)$ and $O(n) \supset O(n-1)$[15], one obtains the values of the angular momentum operators L and their multiplicity β. Hence, this reduction explicitly depends on the the $l-$ orbits appropriate for the nucleus under consideration and could be applied in one or several orbits.

Now, we turn to another important aspect of this reduction, namely its relation to the description of pairing phenomena in the framework of the shell-model algebras. It is proven in Ref. [16] that the $SO(\Omega)$, appearing in the $U(\Omega)$ representation space, restricted by the condition to be in a direct product with the $SU_{ST}(4)$ algebra, is complementary to the $SO(8)$ algebra. On the other hand, the $SO(8)$ is the spectrum generating algebra for the isoscalar ($T = 0$) and isovector ($T = 1$) pair creation and annihilation operators within the nuclear shell model:

$$S^\dagger_\mu = \sum_l \beta_l \sqrt{\frac{2l+1}{2}} \left(a^\dagger{}_{l\frac{1}{2}\frac{1}{2}} \times \tilde{a}^\dagger_{l\frac{1}{2}\frac{1}{2}} \right)^{(010)}_{(0\mu 0)} \tag{11}$$

and

$$P^\dagger_\mu = \sum_l \beta_l \sqrt{\frac{2l+1}{2}} \left(a^\dagger{}_{l\frac{1}{2}\frac{1}{2}} \times \tilde{a}^\dagger_{l\frac{1}{2}\frac{1}{2}} \right)^{(001)}_{(00\mu)} \ , \tag{12}$$

where $\beta_l = +1$ or -1, and the bracket denotes coupling in L, S, T. Obviously, these operators can be expressed

in terms of the Wigner's $SU_{ST}(4)$ generators. Physically relevant reductions of the $SO(8)$ algebra that conserve spin and isospin are realized through three dynamical symmetries[11, 12]:

$$\begin{array}{ccccc} & & SO(8) & & \\ & \swarrow & \downarrow & \searrow & \\ SO_{ST}(6) & & SO_T(5)\otimes SO_S(3) & & SO_S(5)\otimes SO_T(3)\ , \\ & \searrow & \downarrow & \swarrow & \\ & & SO_T(3)\otimes SO_S(3) & & \end{array} \tag{13}$$

which are used to obtain analytical solutions for either isoscalar or isovector pairing interactions in the middle and right side chains respectively or in the left one for both of them with equal strengths, since the $SO_{ST}(6)$ algebra is the same as the one isomorphic to the Wigners' $SU_{ST}(4)$ in Eq. (2). Because of this, they both are labeled by the same quantum numbers: $v,[p_1,p_2,p_3]$ or $[\mu]=[\mu_1,\mu_2,\mu_3,\mu_4]$ with $\mu_1\geqslant\mu_2\geqslant\mu_3\geqslant\mu_4\geqslant 0$, where $v=\sum_i\mu_i$ is the seniority quantum number for $SO(\Omega)$ and $SO(8)$. In the latter case, using the relations for the Casimir invariants of the algebras in Eqs.(13), (2) and (3), the total pairing interaction:

$$V_{P(\text{total})}=\boldsymbol{G}(S^\dagger\cdot S+P^\dagger\cdot P) \tag{14}$$

is diagonal in the basis states of the reduction chain (10) to the angular momentum subgroup $SO_L(3))$ which are labeled as

$$|\Psi_{P(\text{total})}\rangle\equiv|\{f\},v[p_1,p_2,p_3],\beta LM\rangle\equiv|m,v[p],\beta LM\rangle\ , \tag{15}$$

where β gives the multiplicity of the $SO(3)$ representations L appearing in the $SO(8)$ ones. The eigenvalues of the total interaction Eq. (14) in its eigenbasis Eq. (15) are given in Ref. [16]. They do not depend explicitly on L,S and T. In the case of the reduction chain in the middle of the scheme (13), the relevant interaction is the isoscalar pairing:

$$V_{P(\text{isosc})}\equiv V_0=\boldsymbol{G}_0S^\dagger\cdot S\ . \tag{16}$$

The right reduction chain in (13) corresponds to the isovector pairing:

$$V_{P(\text{isov})}\equiv V_1=\boldsymbol{G}_1P^\dagger_\mu\cdot P_\mu\ . \tag{17}$$

Further, we present two other chains of the number-conserving shell- model algebra $U(4\Omega)$ that correspond to the middle and right chains in (13) describing the pure isoscalar Eq. (16) and isovector Eq. (17) pairing. As shown in Ref. [16], since in these cases one has to start with the $U(4\Omega)\supset U(2\Omega)\otimes SU_\sigma(2)$ (4) and $U(2\Omega)$ also contains $SO(\Omega)\otimes SU_\sigma(2)$, where $\sigma=S\vee T$, for completing the group chain the algebra $Sp(2\Omega)$ has to be included in $U(2\Omega)$ and then reduced to $SO(\Omega)\otimes SU_\sigma(2)$. Therefore, the group-subgroup chains equivalent to the middle and right chains in (13) and contained in the number-preserving shell-model symmetry algebra $U(4\Omega)$ are:

$$\begin{array}{llll}[U(2\Omega)\supset & Sp(2\Omega)\supset & SO(\Omega)\otimes SU_S(2)] \\ \{f_s\} & \langle\tilde{\mu}_S\rangle;\upsilon_S\,t_S & v[p] & S\ .\end{array} \tag{18}$$

and

$$\begin{array}{llll}[U(2\Omega)\supset & Sp(2\Omega)\supset & SO(\Omega)\otimes SU_T(2)] \\ \{f_s\} & \langle\tilde{\mu}_T\rangle;\upsilon_T\,t_T & v[p] & T\ .\end{array} \tag{19}$$

Respectively, each of the algebras in Eqs. (18) and (19) is in direct product with the $SU_T(2)$ for the isoscalar and $SU_S(2)$ for the isovector pairing respectively. Hence, the irreps $\{f_\sigma\}=\{2^{f_2},1^{f_1}\}$, where $f_1+f_2=m,f_1\geqslant f_2\geqslant\geqslant 0$ and $\sigma=\frac{f_1-f_2}{2}$ of $U(2\Omega)$, are obtained from the partitions of $U_\sigma(2)$. The quantum numbers of the irreps $\{\tilde{\mu}_\sigma\}=\{2^{\mu_1},1^{\mu_2}\}$, where $<\upsilon_\sigma=2\mu_1+\mu_2,t_\sigma=1/2\mu_2>$ of $Sp(2\Omega)$[16], are introduced in analogy to the seniority and reduced spin/isospin. The equivalence of the reduction chains (18) and (19) and the middle and right chains in (13) describing the pure isoscalar (16) and isovector (17) pairing respectively, as established in Ref. [16] is due to the complementarity introduced in Ref. [17] and in this case is between $Sp(2\Omega)$ and $SO(8)$ as well as $SO(\Omega)\otimes SU_\sigma(2)$ and $SO(5)\otimes SO_\sigma(3)$. Then, the basis states for the Hamiltonians V_0 (16) and V_1 (17) in the reductions (18) and (19) to the angular momentum algebra $SO(3)$ are labeled as

$$|\Psi_{P(\text{isosc})}\rangle\equiv|m,S,\upsilon_S,t_S,T,v[p]\beta LM\rangle\ , \tag{20}$$

and

$$|\Psi_{P(\text{isov})}\rangle\equiv|m,T,\upsilon_T,t_T,S,v[p]\beta LM\rangle\ , \tag{21}$$

and the interactions given by Eqs. (16) and (17) are diagonal in the isoscalar Eq. (20) and isovector Eq. (21) pairing bases with eigenvalues given in Ref. [16]. Therefore, the isovector chain (19) generates rotations $T(T+1)$ in the isospin space and in this way differs from the chain (18). Due to relations between the Casimir invariants of the subgroups in the chains (18), (19)[16], the eigenenergies of Eqs. (16) and (17) do not explicitly depend on the quantum numbers $v[p]$ of the representations $SO(\Omega)\sim SO(5)$, but instead depend on the quantum numbers υ_S,t_S and υ_T,t_T respectively, which define the representations of $Sp(2\Omega)\supset SO(\Omega)$ in each of the considered cases. By using the standard methods for the decomposition $O(n)\supset O(n-1)$[15],

one obtains the values of the angular momentum operators L and their multiplicity β. Hence, this reduction explicitly depends on the $l-$ orbits appropriate for the nucleus under consideration and could be applied in one or several orbits.

3 Unified reduction schemes in the shell-model algebra

Unifying the reductions of the shell-model algebra $U(4\Omega)$ (1) in the spatial and spin-isospin branches and taking into account the complementarity of the reduction of the spatial part $U(\Omega)$ and the spin-isospin part of the Wigner's supermultiplet model (2)[10], as well as the two possible realizations of the reductions of the spatial part, namely through the $SU(3)$ algebra (5)[7] and through the $SO(\Omega)\Leftrightarrow SO(8)$ (10)[16], we can unify the considered above chains into a generalized reduction scheme of the type.

Now, we obtain the important result that the spatial subalgebra $U(\Omega)$ of the shell-model algebra $U(4\Omega)$ contains two distinct dynamical symmetries, defined by the reduction chains - left branch (10) and right branch (5). Both of them are complementary to the Wigner's supermultiplet (2). It is clearly seen in the Fig. 1, that there is a horizontal correspondence of the irreps of $SO(\Omega)\sim SO(8)$ and the ones of $SU(3)$.

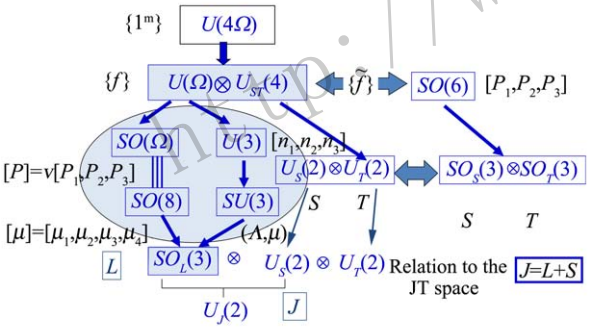

Fig. 1 (color online) Unified reduction scheme of $U(4\Omega)$ with the correspondence between the total pairing chain and the $SU(3)$ chain.

As shown in the previous section, the reduction of the $SO(8)$ algebra, complementary to the $SO(\Omega)$ shell-model algebra, can be completed by means of reducing the representations of $U(2\Omega)$ into the ones of the symplectic algebra $Sp(2\Omega)$ and then going down to $SO(3)$ algebra through the direct product $SO(\Omega)\otimes SU(2)$. The later $SU(2)$ algebra determines the isoscalar (16) or isovector (17) character of the corresponding pairing interaction. Below in Fig. 2, we illustrate the unified reduction scheme for the isoscalar case. The isovector case is obtained by exchange of the labels S and T in the coresponding $SU(2)-$algebras and their representations.

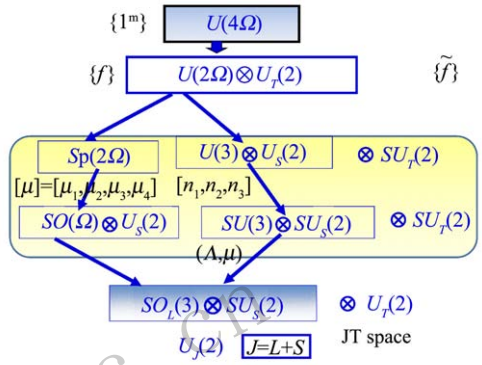

Fig. 2 (color online) Unified reduction scheme for the isoscalar pairing chain and the $SU(3)$ chain.

Here again, the horizontal correspondence of the irreps of $Sp(\Omega)\sim SO(5)$ and the ones of $SU(3)$ is clearly observed. From the above generalized reduction schemes it could be seen that the chains defining the reductions in them on Figs. 1 and 2 determine full-basis sets in which the basis states could be classified and the correspondence between the representations labeling them in the pairing and rotational bases could be explicitly obtained. We illustrate this on Tables 1 and 2 for the case of 4 particles in the *ds* shell. This is not the general case, for which the reductions given on Figs. 1 and 2 are valid, but we choose it, since it will be presented in the applications for the ^{20}Ne nucleus. The isovector result can be obtained by interchanging the S and T labels. From the Tables 1 and 2 could be clearly seen that the $SU(3)-SO(8)$ correspondence restricts which $SU(3)$ irreps are present in a certain pairing eigenstate. Although slightly changed, the picture remains almost the same in the isoscalar case but only slightly expands the number of $SU(3)$ irreps that are allowed.

Table 1 The classification of the states of 4 particles in the *ds* shell ($\Omega=6$) according to the reduction scheme on Fig. 1.

$U(6)$ $\{f\}$	$SO(6)$ $[\tilde{\mu}][\mu]$	$SO_p(6)$ $\nu[p]$	$SU(3)$ (λ,μ)	K	$SO(3)$ L	$U_{ST}(4)$ $\{\tilde{f}\}$	$SO_P(6)$ $[P]$	$SU_S(2)\times SU_T(2)$ (ST)
$\{1^4\}_{15}$	$[1^2]_{15}[2]$	$2[1^3]$	$(1,2)_{15}$	1	1,2,3	$\{4\}_{35}$	$[2^3]_{35}$	$(0,0)_1$
								$(1,1)_9$
								$(2,2)_{25}$

Table 1 (Continued)

$U(6)$ $\{f\}$	$SO(6)$ $[\tilde{\mu}][\mu]$	$SO_p(6)$ $\nu[p]$	$SU(3)$ (λ,μ)	K	$SO(3)$ L	$U_{ST}(4)$ $\{\tilde{f}\}$	$SO_P(6)$ $[P]$	$SU_S(2)\times SU_T(2)$ (ST)
$\{21^2\}_{105}$	$[211]_{45}[31]$	$4[21^2]$	$(0,1)_3$	0	1	$\{31\}_{45}$	$[21^2]_{45}$	$(1,0)_3$
	$[21-1]_{45}$		$(2,3)_{42}$	0	1,3,5			$(0,1)_3$
				2	2,3,4			$(2,1)_{15}$
			$(5,0)_{21}$	0	1,3,5			$(1,2)_{15}$
			$(3,1)_{24}$	1	1,2,3,4			$(1,1)_9$
	$[1^2]_{15}[2]$	$2[1^3]$	$(1,2)_{15}$	1	1,2,3			
$\{2^2\}_{105}$	$[0]_1[0]$	0[0]	$(0,4)_{15}$	0	0,2,4	$\{2^2\}_{20}$	$[2]_{20}$	$(2,0)_5$
	$[2]_{20}[1^2]$	2[1]	$(2,0)_6$	0	0,2			$(1,1)_9$
	$[2^2][2^2]$	4[2]	$(4,2)_{60}$	0	0,2,4,6			$(0,2)_5$
				2	2,3,4,5			$(0,0)_1$
			$(3,1)_{24}$	1	1,2,3,4			
$\{31\}_{210}$	$[1^2]_{15}[2]$	$2[1^3]$	$(1,2)_{15}$	1	1,2,3	$\{21^2\}_{15}$	$[1^2]_{15}$	$(1,0)_3$
	$[2]_{20}[1^2]$	2[1]	$(6,1)_{63}$	1	1,2,3,4,5, 6,7			$(0,1)_3$
	$[31]_{175}[21^2]$	$4[1^2]$	$(4,2)_{60}$	0	0,2,4,6			$(1,1)_9$
				2	2,3,4,5			
			$(2,3)_{42}$	0	1,3,5			
				2	2,3,4			
			$(3,1)_{24}$	1	1,2,3,4			
			$(2,0)_6$	0	0,2			
$\{4\}_{126}$	$[4]_{105}[1^4]$	4[0]	$(8,0)_{45}$	0	0,2,4,6,8	$\{1^4\}_1$	$[0]_1$	$(0,0)_1$
			$(4,2)_{60}$	0	0,2,4,6			
				2	2,3,4,5			
	$[0]_1[0]$	0[0]	$(0,4)_{15}$	0	0,2,4			
	$[2]_{20}[1^2]$	2[1]	$(2,0)_6$	0	0,2			

Table 2　The classification of the states of 4 particles in the *ds* shell ($\Omega=6$) according to the isoscalar reduction scheme (Fig. 2). In the second column, the $Sp(12)-$irreps $<\tilde{\mu_1},\tilde{\mu_2}>$ to which correspond $<\upsilon_s,t_s>$ are listed. In the forth column, $\{\tilde{f}\}_6\supset\oplus(\lambda,\mu)$ the groups of the corresponding $SU(3)$ irreps are listed by the $SU(6)$ irreps from Table 1 that contain them. The groups separated by commas correspond to the respective spins S from the previous column.

$U(12)$ $\{f\}$	$Sp(12)$ $<\tilde{\mu}>$	$SO(6)\times SU_S(2)$ $[\mu]\times S$	$U(6)\supset\oplus SU(3)$ $\{\tilde{f}\}_6\supset\oplus(\lambda,\mu)$	$U_T(2)$ $\{\tilde{f}\}$	$SU_T(2)$ T
$\{1^4\}_{495}$	$<1^4>$	$[2^2]\times 0$	$\{2^2\}$	$\{4\}$	2
		$[1^4]\times 2$	$\{1^4\}$		
		$[21^2]\times 1$	$\{21^2\}$		
		$[21-1]\times 1$	$\{21^2\}$		
	$<1^2>$	$[2]\times 0$	$\{2^2\}$		
		$[1^2]\times 1$	$\{21^2\}$		
	$<0>$	$[0]\times 0$	$\{2^2\}$		
$\{21^2\}_{2145}$	$<21^2>$	$[2]\times 1$	$\{2^2\}+\{31\}$	$\{31\}$	1
		$[1^2]\times 0,1,2$	$\{21^2\}+\{31\},\{1^4\}+\{21^2\}+\{31\},\{21^2\}$		
		$[31]\times 0,1$	$\{31\},\{31\}$		
		$[21^2]\times 0,1,2$	$\{21^2\},\{21^2\},\{21^2\}$		
		$[21-1]\times 0,1,2$	$\{21^2\},\{21^2\},\{21^2\}$		
		$[2^2]\times 1$	$\{2^2\}$		
		$[1^4]\times 1$	$\{1^4\}+\{21^2\}+\{31\}$		

Table 2 (Continued)

$U(12)$ $\{f\}$	$Sp(12)$ $<\tilde{\mu}>$	$SO(6)\times SU_S(2)$ $[\mu]\times S$	$U(6)\supset\oplus SU(3)$ $\{\tilde{f}\}_6\supset\oplus(\lambda,\mu)$	$U_T(2)$ $\{\tilde{f}\}$	$SU_T(2)$ T
	$<2>$	$[0]\times 1$	$\{2^2\}$		
		$[2]\times 1$	$\{2^2\}+\{31\}$		
		$[1^2]\times 0$	$\{2^2\}+\{31\}$		
	$<1^2>$	$[2]\times 0$	$\{31\}$		
		$[1^2]\times 1$	$\{1^4\}+\{2^2\}+\{31\}$		
$\{2^2\}_{1716}$	$<2^2>$	$[0]\times 0,2$	$\{2^2\}+\{4\},\{2^2\}$	$\{2^2\}$	0
		$[2]\times 0,1,2$	$\{2^2\}+\{4\},\{31\},\{2^2\}$		
		$[1^2]\times 1$	$\{21^2\}+\{31\}$		
		$[31]\times 1$	$\{31\}$		
		$[21^2]\times 1$	$\{21^2\}$		
		$[21-1]\times 1$	$\{21^2\}$		
		$[2^2]\times 0,2$	$\{2^2\},\{2^2\}$		
		$[1^4]\times 0,1$	$\{1^4\},\{21^2\}+\{31\}$		
		$[4]\times 0$	$\{4\}$		
	$<1^2>$	$[2]\times 0$	$\{2^2\}+\{4\}$		
		$[1^2]\times 1$	$\{21^2\}+\{31\}$		
	$<0>$	$[0]\times 0$	$\{2^2\}+\{4\}$		

4 Transformation brackets between the pairing and quadrupole dynamical symmetries

From the above generalized reduction schemes it could be seen that both chains defining the reductions in the spatial part of the generalized schemes in Figs. 1 and 2 determine full-basis sets and could be expressed through each other. Since the microscopic $SU(3)$ model based on the three-dimensional harmonic oscillator has a well-developed theory, including the Wigner-Racah algebra for the calculation of matrix elements[18, 19] in the $SU(3)$ basis and various successful applications in real nuclei[20], we choose to expand the states of the pairing bases (15), (20) and (21) with the quantum numbers $\{\upsilon[p]\beta\}$, $\{\upsilon_S,t_S,\beta\}$ and $\{\upsilon_T,t_T,\beta\}$ correspondingly, labeled by the set of numbers $\{i\}$ through the set of basis states (9) with the quantum numbers $\{\alpha(\lambda,\mu)K\}$ denoted as the set $\{j\}$:

$$|\Psi_P\rangle_i\equiv|\{f\},i,L,M,S,T\rangle=\sum_j C_{ij}|\{f\},j,L,M,S,T\rangle\ . \tag{22}$$

As a result of the dynamical symmetry, the pairing interactions Eq. (14), (16) and (17) are diagonal in the pairing bases (15), (20) and (21) with eigenvalues given in Ref. [16]. Using the expansion (22) in the $SU(3)$ basis states and the diagonalization procedure for the matrices of the pairing interactions in the $SU(3)$ basis:

$$<\Psi_P|H_{\rm pair}|\Psi_P>=E_{\rm pair}(m,i,[P],(ST))=\sum_{jk}C^*_{ki}C_{ij}.\delta_{kj\cdot k}<\Psi_R|H_{\rm pair}|\Psi_R>_j\ , \tag{23}$$

we obtain numerically the probability $|C_{ij}|^2$ with which the states of the $SU(3)$ basis enter into the expansions of the pairing bases Eq. (22). In this way, we actually calculate the transformation brackets between each two chains[21] of the pairing and quadrupole interactions. This is of great use when calculating the matrix elements of different operators in each of the chains. In particular, since we do not have an explicit representation of the pairing bases in terms of the fermion creation and annihilation operators, we can use the transformation brackets to calculate different matrix elements in it. This is important for example for the calculation of transition probabilities. Also, this expansion could help evaluate the importance (weight) of the different $SU(3)$ - states, when we need to impose restrictions on the basis because of computational difficulties. The known relations of the $SU(3)$ labels (λ,μ) and the β,γ shape variables of the geometrical model can be used for the analysis of the deformations of the pairing states, expressed through the respective $SU(3)$ ones.

We illustrate the above results in Fig. 3, where the bars correspond to the distribution in percentages of the $SU(3)$ irreps within the first five pairing states

-$L = 0_1^+, 2_1^+, 4_1^+$ of the ground state band and the band-heads $0_2^+, 2_2^+$ of the quasi$-\beta$ with $K^\pi = 0^+$ and quasi$-\gamma$ with $K^\pi = 2^+$ respectively, for the system of 2 protons and 2 neutrons with $S = 0, T = 0$. From the energy values and the distribution of $SU(3)$ irreps in the pairing states one could reconstruct the structure of the collective bands in the obtained classification. Even only the first five low - lying states given on Fig. 3, show that the bands are constructed from pairing states with all possible seniorities, not just seniority 0, which is very often the case of the shell- model calculations. Also, the leading $SU(3)$ irreps with the highest weight in its containing pairing state give a clear indication for its shape or deformation.

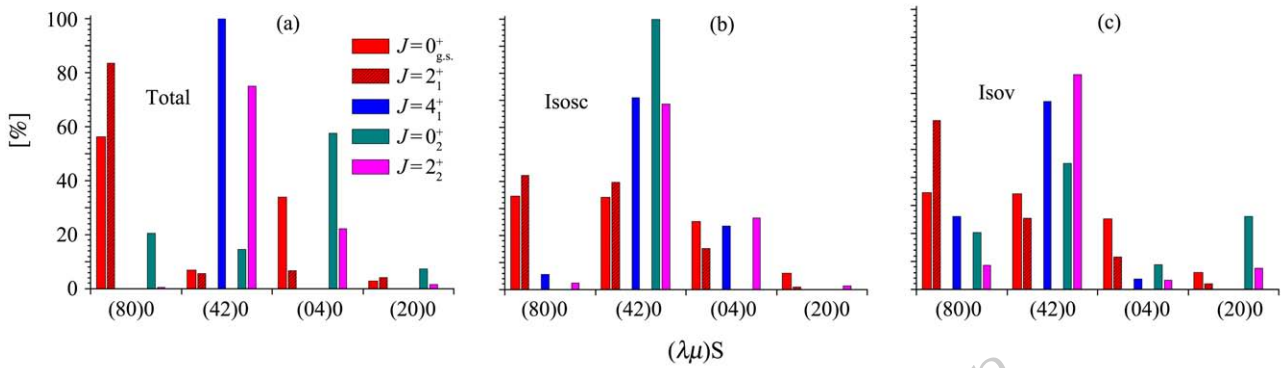

Fig. 3　(color online) The distribution of the $SU(3)$ irreps in the lowest-lying pairing states $L = 0^+_{g.s.}, 2^+_1, 4^+_1$ and $0^+_2, 2^+_2$ for the system of 2 protons and 2 neutrons with $S = 0, T = 0$. (a) The result for the total pairing, (b) the isoscalar case and (c) the isovector case.

5　Calculations and results

As an application of the theory given in the preceding sections, we present the simple case of 4 particles in the *ds* shell (see Tables 1. and 2.) and in particular the $N = Z$ even-even nucleus ^{20}Ne. For its energy spectra we evaluate the interaction strengths of each of the total, V_{total}, Eq. (14), isoscalar, V_0, Eq. (16), isovector, V_1, Eq. (17) and quadrupole-quadrupole (8) interactions in the correct reproduction of the experimental energies[22], and the root mean squared /RMS/ deviation $\sigma = \sqrt{\sum_i (E^i_{\text{Th}} - E^i_{\text{Exp}})^2 / d}$ (per degree of freedom d) of the model energies from the experimental ones. First, we consider the two parameters case, and then aim to improve it by considering the three-parameter one.

The Hamiltonian we use for studying the energy spectrum of the considered realistic nuclear systems in the case of two parameters, can be writtem as:

$$V_{\text{res}} = \frac{1}{2}(1-x)V_k + \frac{1}{2}(1+x)V_2 \ , \qquad (24)$$

where x is a control parameter, which obviously at $x = 1$ redices to the pure $V_2 = \frac{\chi}{2} Q \cdot Q$ interaction and at $x = -1$ gives the limiting case of pure V_k pairing interactions $V_k = V_{\text{P(total)}}(14)$, $V_k = V_0(16)$ and $V_k = V_1(17)$. Hence, we investigate the influence of each of the considered pairing interactions in conjunction with the quadrupole-quadrupole interaction. So, compared to the earlier $SU(3)$ one-shell realization[23, 24] of the PQM, we use a more general pairing Hamiltonian which includes proton-neutron pairing terms as well.

In Fig. 4(a), we present the results of a minimization procedure for the root-mean-squared /RMS/ value σ with respect to the two parameters' fits of $\boldsymbol{G}$, $\boldsymbol{G}_0$ and $\boldsymbol{G}_1$ versus χ of the residual interactions (24) for ^{20}Ne, performed over the 21 lowest-lying positive-parity experimental energies E^i_{Exp}[22]. The lines connecting their limiting values at $x = 1$ (pure $Q{\cdot}Q$ interaction) and $x = -1$ (only one of the pairing interactions) actually represent the intervals of change of each of the parameters. The middles of these lines at $x = 0$ show the values of the parameters, where both interactions are with equal strengths. From Fig. 4(a), it could be seen that for ^{20}Ne the parameter χ is prevailing, introducing a deformation of nuclear shape, and rotational like spectra. But also it is clear that the pairing interactions could not be neglected and in particular the total pairing should be taken into account.

The figures (b), (c) and (d) in Fig. 4, show the change of the energy values of the first five low lying states given on Fig. 2 for x varying in the interval $[-1, 1]$ in the three considered cases of correlations between the total, isoscalar and isovector pairing and the quadrupole-quadrupole interactions. It could clearly be seen that the strong degeneracies in the pairing limits are almost immediately removed from the quadrupole-quadrupole interaction, which introduces

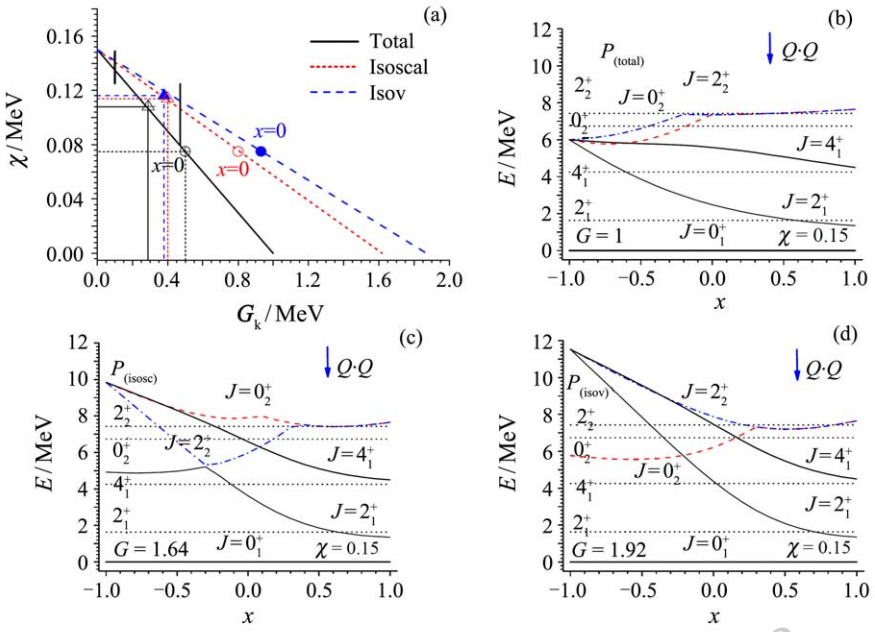

Fig. 4 (color online) Results for the nucleus ^{20}Ne with the Hamiltonians (24). (a) The values of the parameters χ and $\boldsymbol{G}$, $\boldsymbol{G}_0$ and $\boldsymbol{G}_1$ between the limits of pure pairing and quadrupole interactions, defining the intervals of change of the control parameter x. On each of the three lines with a triangular shape are given the values of the two parameters $\boldsymbol{G}_k, \chi$ where σ is minimal. The dots are the values of $x = 0$. (b) The behavior of the excitation energies of the lowest-lying pairing states as functions of the control parameter x varying from -1 to 1 along the black dashed lines. The labels in italic and the dotted lines represent the experimental energies[22].

the mixing of their corresponding $SU(3)$-irreps (see also the Tables 1 and 2). The interesting conclusion from these two-parameter figures for the ^{20}Ne is that for different values of $-1 < x < 1$ different types of spectra, varying from purely rotational to somewhat more vibrational-like can be obtained.

Further, we can separate the two pairing modes - the isoscalar and the isovector one - and use a Hamiltonian:

$$H = V_{\mathrm{P(isosc)}} + V_{\mathrm{P(isov)}} - \frac{\chi}{2} Q \cdot Q \ . \qquad (25)$$

In this case, one has to introduce two control parameters y and z, described in detail in Ref. [8]. These are defined as having the following relation with the three strengths $\boldsymbol{G}_0$, $\boldsymbol{G}_1$ and χ of the Hamiltonian (25): $y = \chi/(\chi+G_1)$, $z = (\chi+G_1)/(\chi+G_0+G_1)$ and the scaling parameter $c = \chi + G_0 + G_1$. Using them, the Hamiltonian becomes $H = c(1-z)V_{\mathrm{isosc}} + c(1-y)zV_{\mathrm{isov}} - cyzQ\cdot Q$. The best three-parameter results, presented in Fig. 5 are obtained for the values $\chi = 0.108$ MeV, $\boldsymbol{G}_0 = \boldsymbol{G}_1 = 0.29$ MeV. In this case the addition of a third parameter does not much change the quality of description of the experimental results. The values of the two pairing strengths are almost equal and the value of the rotational interaction slightly diminishes. This effect could be due to the small dimension $\Omega = 6$ of the ds-shell. It must be pointed out that the quality of the fit also depends on the identification of the experimentally ob-

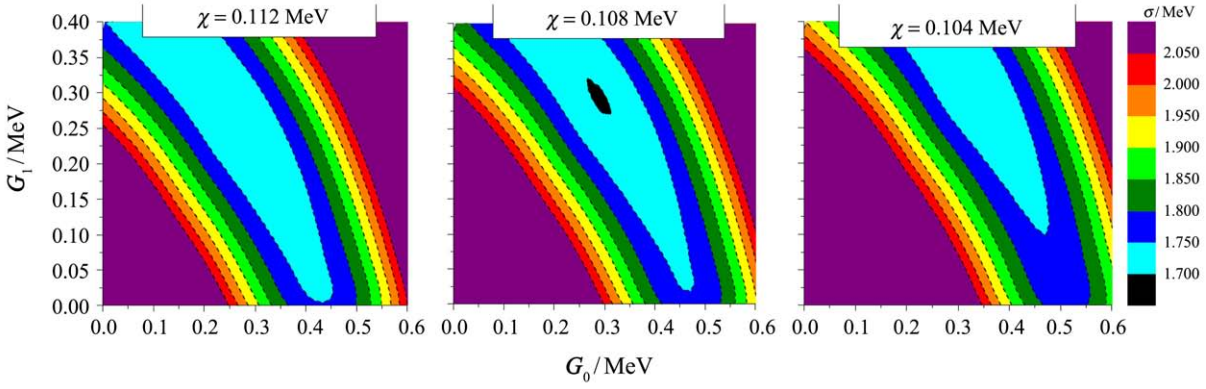

Fig. 5 (color online) The values of the RMS σ, in colors, as functions of the parameters $\boldsymbol{G}_0$ and $\boldsymbol{G}_1$ for different values of χ.

served states with states of the pairing bases, which is not unique, due to their multiple degeneracies. This problem could partially be solved by calculating the transition probabilities, which is not a problem in this approach and is our further aim.

There is also a possibility in this approach to calculate systematically the other types of nuclei with 4 valence particles from which we have *e.g.* 4, 3 and 1 protons and see what type of pairing and ordering of the collective states appear in comparison with this case of equal number of protons and neutrons.

6 Conclusions

On the basis of the algebraic reductions of the spatial part of the shell-model algebra $U(4\Omega)$ the dynamical symmetries of the microscopic pairing algebras, containing the isoscalar ($T = 0,\ S = 1$), the isovector ($T = 1,\ S = 0$), and total (with both of them with equal strengths) pairing interactions[16] are investigated in the framework of Elliott's $SU(3)$ algebra, which is also present there and identifies the rotational limit of the shell model[7]. These reduction chains elucidate the algebraic structure of an extended Pairing-plus-Quadrupole Model. From the Pauli principle it follows that the spatial part is conjugated to the spin-isospin part of the shell model wave function. Hence there exists a complementarity of the representations of the reductions groups of the spatial part and the Wigner supermultiplet ones. This leads to interrelations between the respective irreps of the pairing algebras and the $SU(3)$ one. This is an important result, that allows us to study the complementarity and competitive effects of the quadrupole-quadrupole and pairing interactions on the energy spectra of the nuclear systems. The probability distributions or transformation brackets with which the states of the $SU(3)$ basis enter into the expansion of the pairing basis are obtained numerically. The parameter adjustment for realistic nuclear systems gives the influence of each of the considered pairing and quadrupole modes on the reproduction of the nuclear spectra. This evaluation is achieved and clarified by the introduction of two or three control parameters. For illustration the theoretical results are compared with experimental energy spectra of the nucleus ^{20}Ne, from where the optimal values of two and three parameters of the residual interactions are obtained. A further and natural development of this model is its realization in more than one shell. Some steps have already been done in this direction[9] but one should be cautious with the choice of nuclei to be studied and the model spaces to work in.

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$SO(8)$对关联和代数壳模型中的$SU(3)$四极基底

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摘要： 建立了 $SO(8)$ 同位旋标量、同位旋矢量及总的配对基与微观壳模型坐标空间部分的 Elliott $SU(3)$ 基之间的对应关系。从该代数间的互补关系导出了在壳模型的粒子数守恒代数 $U(4\Omega)$ 中所包含的具有同位旋 T 及自旋 S 的 Wigner 超多重态 (不可约) 表示。其重要性在于，该结果能用于研究对相互作用和四极-四极相互作用在核谱中的竞争效应并揭示其配对基中的 $SU(3)$ 组份。虽然仅展示了该理论对 ds 壳的计算，其方法也适用于研究多壳的情形。

关键词： 代数模型；对相互作用；四极-四极相互作用；壳模型；自旋-同位旋空间

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