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Vortex Solutions of Bilayer Quantum Hall Systems at $\upsilon = 1/2$

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Abstract: We investigate the static vortex solutions of a bilayer quantum Hall state at the Landau-level filling factor $\upsilon = 1/2$. This work is based on the ZHK model, which is an effective field theory including Chern-Simons gauge interactions. We deduce the dimensionless nonlinear equations of motion for vortices possessing cylindrically symmetry, and analyze the asymptotical behaviors of solutions. Additionally, we analyze the values of critical coupling constants under the self-dual condition, and obtain the self-dual equations. Finally, vortices of type $(0,\ 1)$, $(0,\ -1)$, $(1,\ -1)$ and $(-1,\ -1)$ are solved with numerical methods. We reach the conclusion that vortex of type $(1,\ -1)$ is unstable, which will decay to $(1,\ 0)$ and $(0,\ -1)$. The vortices of type $(0,\ -1)$ and $(-1,\ -1)$ are self-dual solutions from numerical results.

Key words: bilayer quantum Hall system; Chern-Simons gauge interaction; self-dual condition; vortex

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1 Introduction

The discovery of the fractional quantum Hall effect (FQHE) in 1982 opened an exciting new area in condensed matter physics. Many theories have been constructed to explain the FQHE, among of which the ZHK model is a simple and effective one. The ZHK model, proposed by Zhang et al.[1−2] and Read[3], is an effective field theory like the Landau-Ginzburg theory of superconductivity. It could describe almost all the properties of the fractional quantum Hall effect. In this model, the interacting electrons are not only coupled to an external electromagnetic field but also interacted with an additional gauge field, which is the Chern-Simons gauge field.

Bilayer quantum Hall systems are typically constructed by trapping electrons in two thin layers at the interface of the semiconductors. This structure introduces an additional degree of freedom in the z direction, which means new physics different from the monolayer systems. In this paper, we will study a special type of FQH states, which is called the bilayer-locked state, denoted by $(m,\ n,\ l)$[4]. Here, m, n should be two odd integers, and l be an integer, $l \leqslant m,\ n$. The electrons between the two layers are strongly correlated, which could be described by the Halperin wave function. We will search for the vortex solutions in a bilayer-locked state with the Landau-level filling factor $\upsilon = 1/2$, which is denoted by $(3,\ 3,\ 1)$. This state can only exist in bilayer systems, and has been observed experimentally [5−6].

2 The ZHK model in bilayer systems

We consider electrons confined in the xy plane in the presence of a uniform external magnetic field along the z axis, $B = (0,\ 0,\ -B_\perp)$, with $B_\perp > 0$. Provided that the Zeeman splitting is large enough and the electrons are all polarized, the dynamical degree of freedom associated with spins could be ignored. Thus electrons in different layers could be described by two scalar fields φ_I. In the following analysis, we take the notation that the charge of electron is $q = -e,\ e = 1$ and the metric is $g_{\mu\nu} = (1,\ -1,\ -1)$.

The Lagrangian of the effective theory of the FQHE

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in bilayer systems takes the form from Refs.[7–9] (in units in which $c = \hbar = 1$)

$$L = \sum_{I=1,\, J=1}^{2} -\Theta_{IJ}\varepsilon^{\mu\nu\lambda} a_{I\mu}\partial_\nu a_{J\lambda} + \sum_{I=1}^{2} \varphi_I^\dagger \left(\mathrm{i}\partial_0 - q(a_{I0}+A_0)\right)\varphi_I - \frac{1}{2M}\left|(-\mathrm{i}\nabla + q(\mathbf{a}_I+\mathbf{A}))\varphi_I\right|^2 - V(\rho) \tag{1}$$

with

$$V(\rho) = \sum_{I=1}^{2} \frac{U}{2}(|\varphi_I|^2-\rho)^2 + W(|\varphi_1|^2-\rho)(|\varphi_2|^2-\rho)\ ,$$

where the subscript, $I,\ J = 1,\ 2$, labels the layers, and the Greek index, $\mu,\ \nu,\ \lambda = 0,\ 1,\ 2$, labels the time and space components.

The first term in Eq. (1) is the Chern-Simons term, which is associated with all the topological properties of the system, such as the Hall conductance and the fractional charge carried by the vortex. This model introduces two Chern-Simons gauge fields, denoted by $a_{I\mu}, I = 1,\ 2$. They are coupled by the coupling matrix

$$\Theta = \frac{e^2}{4\pi(m^2-n^2)}\begin{pmatrix} m & -n \\ -n & m \end{pmatrix}$$

which determines the statistical properties of electrons. In this paper, we work with $m = 3$ and $n = 1$. $\varepsilon_{\mu\nu\lambda}$ is the antisymmetric tensor with $\varepsilon_{012} = 1$, the effect of which breaks the the parity P and time reversal invariant T.

The covariant derivative in the next two terms involves both the Chern-Simons gauge field $a_{I\mu}$ and the external gauge field A_μ. A_μ is the external gauge field describing the external magnetic field. As we never consider the electric field here, it is convenient to work with the Coulomb gauge, where $A_0 = 0$ and $\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$.

The last term in Eq. (1) is the self-interaction term, where the long-range Coulomb interaction between electrons is replaced by effective contact interactions[1, 10–11]. The factor $1/2$ here is to ensure that each interaction is included only once. U and W are both positive coupling constants, which have dimensions of length. Generally, intralayer interactions are not equal to interlayer interactions, $U \neq W$[11]. The average density in the two layers takes the same value ρ at the $(3,\ 3,\ 1)$ state from the result[4].

The equations of the motion from the Lagrangian Eq. (1) are

$$q\widetilde{A}_{I0}\varphi_I = \frac{1}{2M}D_{Ii}^2\varphi_I - \frac{\partial V}{\partial\varphi_I^\dagger} \tag{2}$$

with

$$\frac{\partial V}{\partial\varphi_1^\dagger} = U(|\varphi_1|^2-\rho)\varphi_1 + W(|\varphi_2|^2-\rho)\varphi_1\ ,$$

$$\frac{\partial V}{\partial\varphi_2^\dagger} = U(|\varphi_2|^2-\rho)\varphi_2 + W(|\varphi_1|^2-\rho)\varphi_2\ ,$$

$$\sum_{J=1}^{2}\Theta_{IJ}(\partial_1 a_{J2} - \partial_2 a_{J1}) = -\frac{q}{2}|\varphi_I|^2\ , \tag{3}$$

$$\sum_{J=1}^{2}\Theta_{IJ}\varepsilon_{ij}\partial_j a_{J0} = \frac{q}{4M\mathrm{i}}\left\{(D_{Ii}\varphi_I)^\dagger\varphi_I - \varphi_I^\dagger D_{Ii}\varphi_I\right\}. \tag{4}$$

Here, we ignore the terms which possess the derivative of fields with respect to time. Since we are concerned with static classical solutions, all fields are time independent.

The covariant derivative is given by $D_{I\mu} = \partial_\mu + \mathrm{i}q\widetilde{A}_{I\mu}$, with effective vector potential $\widetilde{A}_{I\mu}$ defined by $\widetilde{A}_{I\mu} = a_{I\mu} + A_\mu$. Note that $D_{I\mu}$ involves the Chern-Simons gauge field and the external gauge field, both of which are combined into the effective vector potentials $\widetilde{A}_{I\mu}$. It is natural to express the equations in terms of effective vector potentials $\widetilde{A}_{I\mu}$, instead of $a_{I\mu}$ and A_μ, as in Ref. [12].

From the constraint Eq. (3), the Chern-Simons vector fields are completely determined by the electron densities. It coincides with the fact that the Chern-Simons fields are statistical fields rather than dynamical fields. There are only two independent dynamical fields φ_1 and φ_2.

We can change the form of Eq. (3) into

$$b_1 = \frac{2\pi}{e}\left(m|\varphi_1|^2 + n|\varphi_2|^2\right)\ ,$$

$$b_2 = \frac{2\pi}{e}\left(n|\varphi_1|^2 + m|\varphi_2|^2\right)\ , \tag{5}$$

where b_I are the Chern-Simons magnetic fields, defined by $b_I = \partial_1 a_{I2} - \partial_2 a_{I1}$. It means the electron in the first layer carries two kinds of fluxes, i.e. the flux of $a_{1\mu}$ is $\phi_1 = 2\pi m/e$, and the flux of $a_{2\mu}$ is $\phi_2 = 2\pi n/e$. The situation is similar for the electron in the second layer, which possesses $\phi_1 = 2\pi n/e$ and $\phi_2 = 2\pi m/e$. As can be seen in Eq. (1), the electron in the Ith layer can only couple with the $a_{I\mu}$. Considering two electrons in the same layer, as one electron moves adiabatically around another electron, the wave function acquires a phase change $2\pi m$, due to the Aharonov-Bohm effect. But in the case when electron 1 is in the first layer, electron 2 in the second layer, we move electron 2 round the electron 1. The moving electron 2 can only feel the part of flux $\phi_2 = 2\pi n/e$, carried by electron 1, so the phase change is $2\pi n$. All the conclusions coincide with the Halperin wave function[13]

$$\psi_{mmn} = \prod(z_i - z_j)^m \prod(w_i - w_j)^m \prod(z_i - w_j)^n \times \exp\left[-\frac{1}{4}\left(\sum |z_i|^2 + \sum |w_i|^2\right)\right] . \quad (6)$$

It is necessary to mention that the parameter M in Eq. (1) is not the mass of the electron[8] but a parameter with mass dimension depending on the Coulomb interaction[1, 3, 7]. As has been proposed in Zhang et al.'s paper[1], the effective action takes the same form as the microscopic action, but with a renormalized constant M replacing the bare mass m_e, and an effective contact interaction replacing the nonlocal interaction Coulomb interaction. M, U and W are treated as phenomenological parameters.

3 Dimensionless vortex equations with cylindrical symmetry

We will search for a set of static classical vortex solutions. It is natural to take assumptions that the vortices are cylindrically symmetric. Vortices should be solved with polar coordinates $(r,\ \theta)$.

The form of equations of motion can be expressed more clearly, when the Euler-Lagrange equations are rewritten in terms of effective vector potential $\widetilde{A}_I$ only.

First, we note that

$$\widetilde{A}_{I0} = a_{I0} , \quad (7)$$

as $A_0 = 0$. Then, due to the relation $\partial_1 \boldsymbol{A}_2 - \partial_2 \boldsymbol{A}_1 = -B_\perp$, we could obtain

$$\partial_1 a_{I2} - \partial_2 a_{I1} = B_\perp + (\partial_1 \widetilde{A}_{I2} - \partial_2 \widetilde{A}_{I1}) . \quad (8)$$

Substituting Eqs. (7) and (8) into Eqs. (3) and (4) yields

$$\sum_{J=1}^{2} \Theta_{IJ} \left[(\partial_1 \widetilde{A}_{J2} - \partial_2 \widetilde{A}_{J1}) + B_\perp\right] = -\frac{q}{2}|\varphi_I|^2 , \quad (9)$$

$$\sum_{J=1}^{2} \Theta_{IJ} \varepsilon_{ij} \partial_j \widetilde{A}_{J0} = \frac{q}{4M\mathrm{i}} \left[(D_{Ii}\varphi_I)^\dagger \varphi_I - \varphi_I^\dagger D_{Ii}\varphi_I\right] . \quad (10)$$

As we know, the FQHE can develop a series of plateaux, when the magnetic field takes some special values. The strength of $B_\perp$ is expected to be determined by the mean density and the filling factor. We use the result $B_\perp = 2\pi(m+n)\rho e$ directly[4].

The effective magnetic field in the Ith layer is introduced as $\widetilde{B}_I = \partial_1 \widetilde{A}_{I2} - \partial_2 \widetilde{A}_{I1}$. Eq. (9) can be transformed to be

$$\widetilde{B}_1 = 2\pi\left[m(|\varphi_1|^2 - \rho) + n(|\varphi_2|^2 - \rho)\right] , \quad (11)$$

$$\widetilde{B}_2 = 2\pi\left[n(|\varphi_1|^2 - \rho) + m(|\varphi_2|^2 - \rho)\right] . \quad (12)$$

Eqs. (11) and (12) imply that the effective magnetic fields vanish as the densities of electrons approach the mean value. Integrating Eqs. (11) and (12) over the whole space, it is easy to find the relations between the charges and effective magnetic fluxes carried by the vortex. We take the symmetric gauge where the vector potential has the form $\widetilde{A}_I = \widetilde{A}_{I\theta}(r)\hat{\theta}$, where $\hat{\theta}$ is the unit vector along the azimuthal angle. The effective vector potential $\widetilde{A}_I$ has only θ component, which is a function depending on the radius.

In order to make the Euler-Lagrange equations dimensionless, we need to rescale the variables. The scalar field φ_I could be parameterized as the form

$$\varphi_I = \sqrt{\rho} f_I \mathrm{e}^{\mathrm{i}\lambda_I \theta} , \quad (13)$$

where f_I is selected to be real and dimensionless, λ_1 and λ_2 are two independent integers. As the Lagrangian possesses a $U(1) \times U(1)$ symmetry[8], the group space must be $S^1 \times S^1$. Meanwhile, the vortex is characterized by the boundary condition at infinity, which is identical to a large circle S^1. The topology is determined by the homotopy group $\pi_1(S^1 \times S^1) = \pi_1(S^1) \times \pi_1(S^1)$[14] . Consequently, the vortex is characterized by a pair of integers, λ_1 and λ_2. We indentify the vortex with $(\lambda_1,\ \lambda_2)$.

We also define the dimensionless coordinate variable as $\varepsilon = x/\lambda$, where λ is a characteristic length relative to the size of the vortex. The dimensionless effective vector potential $\hat{A}_I$, defined by $\widetilde{A}_{I\mu} = \hat{A}_{I\mu}/e\lambda$, is introduced to make the covariant derivative have the following form[15]

$$\left(\partial_\mu - \mathrm{i}e\widetilde{A}_{I\mu}\right)\varphi_I = \frac{1}{\lambda}\left(\frac{\partial}{\partial \varepsilon^\mu} - \mathrm{i}\hat{A}_{I\mu}\right)\varphi_I . \quad (14)$$

Here we choose the characteristic length $\lambda = 1/\sqrt{2M\rho U}$. Substituting the above Eqs. (13) and (14) into the Eqs. (2), (9) and (10) yields

$$f_I'' + \frac{1}{r}f_I' - \left(\frac{\lambda_I}{r} - \hat{A}_{I\theta}\right)^2 f_I + \kappa_1 \hat{A}_{I0} f_I + (1 - f_I^2) f_I + \beta(1 - f_J^2) f_I = 0 , \quad (15)$$

(If we take $I = 1$, then $J = 2$, and vice versa)

$$\hat{A}'_{I\theta} + \frac{1}{r}\hat{A}_{I\theta} + \hat{B} - \kappa_2 \sum_{J=1}^{2} K_{IJ} f_J^2 = 0 , \qquad (16)$$

$$\hat{A}'_{I0} + \kappa_3 \sum_{J=1}^{2} K_{IJ} f_J^2 \left(\frac{\lambda_J}{r} - \hat{A}_{J\theta} \right) = 0 , \qquad (17)$$

with $K = \frac{e^2}{4\pi}\Theta^{-1}$, i. e.

$$K = \begin{pmatrix} m & n \\ n & m \end{pmatrix} .$$

In Eqs. (15), (16) and (17), r is the dimensionless radius and $\hat{B}$ is defined by $\hat{B} = e\lambda^2 B_\perp$. The $\beta = W/U$ is a parameter depending on the ratio of the two coupling constants W and U. All the three parameters κ_1, κ_2 and κ_3 are dimensionless, and they are defined by

$$\kappa_1 = \sqrt{\frac{2M}{\rho U}} , \quad \kappa_2 = \frac{\pi}{MU} , \quad \kappa_3 = \frac{2\kappa_2}{\kappa_1} .$$

It should be emphasized that $\hat{A}_{I0}$ is a function of radius only. And f_I, $\hat{A}_{I0}$ and $\hat{A}_{I\theta}$ are all dimensionless fields.

4 Asymptotical conditions

The static energy of the vortex is expressed as

$$E = \int d^2x \sum_{I=1}^{2} \left[\frac{1}{2M} |D_{Ii}\varphi_I|^2 + V(\rho) \right] . \qquad (18)$$

As we know, the vortices are stable objects with finite energy. The energy-finiteness condition imposes a certain boundary condition on the solutions of the fields. This condition requires that each term of the energy density in Eq. (18) vanishes at the point far away from the core of the vortex. It means

$$D_{Ii}\varphi_I \to 0 , \qquad (19)$$

$$V(\rho) \to 0, \text{ as } r \to \infty . \qquad (20)$$

the condition Eq. (20) implies that the fields approach the ground-state value

$$|\varphi_I| \to \sqrt{\rho}, \text{ as } r \to \infty . \qquad (21)$$

Consequently, it follows from Eqs. (11) and (12) that the effective magnetic fields vanish at infinity. The effective vector potentials are pure gauges in these areas, which can be expressed as $\widetilde{\boldsymbol{A}}_I = -\frac{1}{q}\nabla(\lambda_I\theta)$. The asymptotical form is $\varphi_I \to \sqrt{\rho}e^{i\lambda_I\theta}$. We rewrite it with the polar coordinates, and find that

$$\widetilde{A}_r \to 0 , \ \widetilde{A}_{I\theta} \to -\frac{\lambda_I}{qr}, \text{ as } r \to \infty .$$

The covariant derivatives satisfy

$$D_{I\theta}\varphi_I \to 0 , \ D_{Ir}\varphi_I \to 0, \text{ as } r \to \infty .$$

Additionally, the total effective magnetic flux in the Ith layer is quantized

$$\widetilde{\phi}_I = \int d^2x \widetilde{B}_I = \frac{2\pi\lambda_I}{e} . \qquad (22)$$

Consequently, the asymptotical conditions at infinity for the dimensionless solutions are

$$\begin{aligned} &f_I \to 1 , \\ &\hat{A}_{I\theta} \to \frac{\lambda_I}{r} , \\ &\hat{A}_{I0} \to 0 . \end{aligned} \qquad (23)$$

The last condition in Eq. (23) is due to Eq. (2), the right part of which vanishes as φ_I approaches the ground value. We use these asymptotical conditions in Eq. (23) to select the right parameters, when applying numerical methods.

It is convenient to rewrite the static energy with integration over the dimensionless radius

$$\begin{aligned} E = &\frac{\pi\rho}{M}\int_0^{+\infty} dr r \sum_{I=1}^{2}\left[f_I'^2 + (\frac{\lambda_I}{r} - \hat{A}_{I\theta})^2 f_I^2 \right] + \\ &2\pi\lambda^2\rho^2 \int_0^{+\infty} dr r \left[\frac{U}{2}\sum_{I=1}^{2}(f_I^2-1)^2 + \right. \\ &\left. W(f_1^2-1)(f_2^2-1) \right] . \end{aligned} \qquad (24)$$

The asymptotic solutions for the short-distance limit are easily found. The fields can be expanded in terms of radius. It is sufficient to retain only the nontrivial lowest-order terms in the core region. The vortex is labeled by a pair of integers (λ_1, λ_2). We will calculate the vortices of type $(0, 1)$, $(0, -1)$, $(1, -1)$ and $(-1, -1)$. In the case of $\lambda_I \neq 0$, the phase of scalar field on the Ith layer is well defined except the point at the core center. It is a singular point where φ_I should vanish, $\varphi_I(0) = 0$. But when we set $\lambda_I = 0$, this kind of constraint never exists. The finiteness of the effective magnetic fields also implies effective vector potential fields $\widetilde{A}_{I\theta}$ vanish at the origin. Thus, we divide our problems into two cases. In the case $\lambda_1=0,\ \lambda_2=\pm1$ the fields can be expressed as

$$\begin{aligned} f_1(r) &\simeq \eta_1-\frac{1}{4}\left(\kappa_1\sigma_1+1+\beta-\eta_1^2\right)\eta_1 r^2\ , \\ f_2(r) &\simeq \eta_2 r-\frac{1}{8}\left[\kappa_1\sigma_2+1+\beta\left(1-\eta_1^2\right)\right]\eta_2 r^3\ , \\ \hat{A}_{I\theta}(r) &\simeq -\frac{1}{2}\left(\hat{B}-\kappa_2 K_{I1}\eta_1^2\right)r+a_{I3}r^3, \\ \hat{A}_{I0}(r) &\simeq \sigma_I+b_{I2}r^2,\ \text{ as } r\ll 1\ , \end{aligned} \tag{25}$$

where, η_1, η_2, σ_1 and σ_2 are free parameters, which fix all the other coefficients of the Taylor expansions of the fields. In the case $|\lambda_1|=1, |\lambda_2|=1$, the situation is the similar

$$\begin{aligned} f_I(r) &\simeq \eta_I r-\frac{1}{8}\left(1+\beta+\kappa_1\sigma_I\right)\eta_I r^3\ , \\ \hat{A}_{I\theta}(r) &\simeq -\frac{1}{2}\hat{B}r+\frac{1}{4}\kappa_2\sum K_{IJ}\eta_J^2 r^3\ , \\ \hat{A}_{I0}(r) &\simeq \sigma_I-\frac{1}{2}\kappa_3\sum K_{IJ}\eta_J^2\lambda_J r^2,\ \text{ as } r\ll 1\ , \end{aligned} \tag{26}$$

here, η_1, η_2, σ_1 and σ_2 are free parameters too.

5 Self-dual condition

Substituting the Bogomol'nyi decomposition[16]

$$|D_1\varphi|^2+|D_2\varphi|^2=|(D_1\pm\mathrm{i}D_2)\varphi|^2\mp qB|\varphi|^2\pm\varepsilon^{ij}\partial_i J_j\ , \tag{27}$$

where

$$J_j=\frac{1}{2\mathrm{i}}\left[\varphi^\dagger D_j\varphi-\varphi(D_j\varphi)^\dagger\right]\ ,$$

into the energy Eq. (18), and dropping the surface term, we get that

$$E=\int \mathrm{d}^2x\sum_{I=1}^{2}\frac{1}{2M}\left[|D_{I\pm}\varphi_I|^2\mp q\widetilde{B}_I|\varphi_I|^2\right]+V(\rho)\ , \tag{28}$$

where

$$D_{I\pm}=D_{I1}\pm\mathrm{i}D_{I2}\ .$$

We find

$$\begin{aligned} E=&\int \mathrm{d}^2x\sum_{I=1}^{2}\frac{1}{2M}|D_{I\pm}\varphi_I|^2+ \\ &\left(\frac{U}{2}\pm\frac{m\pi}{M}\right)\left(\sum_{I=1}^{2}(|\varphi_I|^2-\rho)^2\right)+ \\ &\left(W\pm\frac{2n\pi}{M}\right)\left(|\varphi_1|^2-\rho\right)\left(|\varphi_2|^2-\rho\right)\pm \\ &\frac{\pi(m+n)}{M}\rho\sum_{i=I}^{2}\left(|\varphi_I|^2-\rho\right)\ . \end{aligned} \tag{29}$$

Now, we choose the minus sign and require that U and W take critical values, $U=2\pi m/M$ and $W=2\pi n/M$, respectively, the coefficients of the second and third term vanish. Using Eqs. (11) and (12), the energy can be written in a simple form

$$\begin{aligned} E=&\int \mathrm{d}^2x\sum_{I=1}^{2}\frac{1}{2M}|D_{I-}\varphi_I|^2- \\ &\frac{1}{4\pi(m+n)M}B_\perp(\widetilde{B}_1+\widetilde{B}_2)\ , \end{aligned} \tag{30}$$

where $\widetilde{B}_I=\partial_1\widetilde{A}_{I2}-\partial_2\widetilde{A}_{I1}$ is the effective magnetic fields.

From Eq. (30), the energy is bounded below by a multiple of the total effective magnetic flux. This bound is saturated by solutions to the first order equations

$$(D_{I1}-\mathrm{i}D_{I2})\varphi_I=0\ . \tag{31}$$

Integrating Eq. (30) over the whole systems, we obtain

$$E=-\frac{\rho}{2M}(\widetilde{\phi}_1+\widetilde{\phi}_2)\ , \tag{32}$$

where $\widetilde{\phi}_I$ is the effective magnetic flux defined in Eq. (22). The energy is proportional to the sum of fluxes between two layers. We will verify that the (0, −1) and (−1, −1) both do give the right energy value coinciding with Eq. (32). The two are indeed self-dual solutions.

We decompose the scalar field φ_I into its phase and magnitude $\varphi_I=\sqrt{\rho_I}\mathrm{e}^{i\omega_I}$. From Eq. (31), the effective gauge fields everywhere away from the zeros of the scalar field can be expressed by the phase and magnitude of the scalar fields[16]

$$q\widetilde{A}_{Ii}=-\partial_i\omega_I+\frac{1}{2}\varepsilon_{ij}\partial_j\ln\rho_I\ . \tag{33}$$

Eqs. (11) and (12) then reduce to nonlinear equations for the densities ρ_I

$$\begin{aligned} \bigtriangledown^2\ln\rho_1 &= 4\pi e^2\left[m(\rho_1-\rho)+n(\rho_2-\rho)\right]\ , \\ \bigtriangledown^2\ln\rho_2 &= 4\pi e^2\left[n(\rho_1-\rho)+m(\rho_2-\rho)\right]\ . \end{aligned} \tag{34}$$

Substituting Eqs. (27), (31), (11) and (12) into Eq. (2), it is easy to find that $\widetilde{A}_{I0}=\widetilde{B}_I/2M$. We could also change the form of Eq. (33) into $f_I'/f_I=e\widetilde{A}_{I\theta}-\lambda_I/r$. Using these relations, it is simple to find that Eq. (34) is equivalent to equations of motion (15), (16) and (17).

6 Numerical solutions

The existence of external magnetic field makes the parity P of the systems broken[17]. So the vortex of type $(0,\ 1)$ must behaves differently from $(0,\ -1)$. In addition, it is easy to find that the Lagrangian Eq. (1) maintains the same form, when we interchange the layer index between two layers. This implies we only need to study the solutions of the types such as $(0,\ 1)$ and $(0,\ -1)$, other types such as $(1,\ 0)$ and $(-1,\ 0)$ could be neglected.

We have to solve these nonlinear coupled Eqs. (15), (16) and (17) with numerical method. Here, we choose the electron mean density $\rho = 1$ and set the mass parameter $M = 1$. Suppose that $U = 2$, $W = 2$, this is the case when the intralayer interaction and interlayer interaction equals. We perform numerical calculations for vortices of the type $(0,\ -1)$, $(0,\ 1)$ and $(1,\ -1)$. The energy of vortex is calculated using Eq. (24), and the charge is obtained through the integration of $f_I^2 - 1$. We find the parameters $\eta_1 \simeq 1.116$, $\eta_2 \simeq 1.311$, $\sigma_1 \simeq -0.409$ and $\sigma_2 \simeq -1.646$ for type $(0,\ -1)$, whose energy is $E_{(0,\ -1)} \simeq 1.685$. The total charge carried by this vortex is $\frac{1}{4}e$, with $Q_1 \simeq -\frac{1}{8}e$ and $Q_2 \simeq \frac{3}{8}e$. The effective magnetic flux is found to be $\widetilde{\phi}_1 \simeq 0$ and $\widetilde{\phi}_2 \simeq -2\pi$. The energy for vortex of type $(0,\ 1)$ is $E_{(0,\ 1)} \simeq 7.507$, whose free parameters are $\eta_1 \simeq 1.244$, $\eta_2 \simeq 3.052$, $\sigma_1 \simeq 2.779$ and $\sigma_2 \simeq 16.178$. The total charge carried by this vortex is $-\frac{1}{4}e$, with $Q_1 \simeq \frac{1}{8}e$ and $Q_2 \simeq -\frac{3}{8}e$. The free parameters for vortex $(1,\ -1)$ are $\eta_1 \simeq 3.777$, $\eta_2 \simeq 1.731$, $\sigma_1 \simeq 20.707$ and $\sigma_2 \simeq 4.016$, and the energy is $E_{(1,\ -1)} \simeq 11.237$. It is neutral, with $Q_1 \simeq -\frac{1}{2}e$ and $Q_2 \simeq \frac{1}{2}e$. Since our solutions satisfy the asymptotical conditions at large scale in Eq. (23), it is expected that the vortices take fractional charge values on different la-yers, which agrees with the result from Refs.[4, 18]. Here, the normalized scalar fields f_I and normalized magnetic fields $B_{\mathrm{eff}I} = \widetilde{B}_I/B_\perp$ are depicted as functions of dimensionless radius r for the vortices of type $(0,\ -1)$, $(0,\ 1)$ and $(1,\ -1)$. See Fig. 1 for details. We find that the energy of vortex $(1,\ -1)$ is larger than the sum of $(0,\ -1)$ and $(1,\ 0)$. In this case, vortex $(1,\ -1)$ is not stable. It will decay to two free vortices, the $(0,\ -1)$ and $(1,\ 0)$, with conserved topological number.

We take $U = 2$, but set $W = 0$, which implies the interaction between electrons in different layers is switched off. The vortices take the same charges as the former case. The free parameters of $(0,\ -1)$ are $\eta_1 \simeq 1.082$, $\eta_2 \simeq 1.356$, $\sigma_1 \simeq 0.285$ and $\sigma_2 \simeq -1.654$, whose energy $E_{(0,\ -1)} \simeq 1.808$. The energy of $(0,1)$ $E_{(0,\ 1)} \simeq 7.609$, with $\eta_1 \simeq 1.205$, $\eta_2 \simeq 3.134$, $\sigma_1 \simeq 2.973$, $\sigma_2 \simeq 16.543$ and $E_{(1,\ -1)} \simeq 11.497$. Once again, vortex $(1,\ -1)$ is also unstable. See Fig. 2. Finally, we consider the self-dual solutions, in which case the coupling constants take $U = 2\pi m/M$ and $W = 2\pi n/M$. See Fig. 3. The solutions for $(0,\ -1)$ and $(-1,\ -1)$ are indeed self dual solutions, whose numerical energy value agrees with Eq. (32). The static energy approximately equal π, $E_{(0,\ -1)} \simeq 3.14$. The charges on the first and second layers are nearly $Q_1 \simeq -\frac{1}{8}e$ and $Q_2 \simeq \frac{3}{8}e$, and the effective fluxes are $\widetilde{\phi}_1 \simeq 0$ and $\widetilde{\phi}_2 \simeq -2\pi$. Similarly, the energy for $(-1,\ -1)$ vortex is almost $E_{(-1,\ -1)} \simeq 2\pi$, and the charges are $Q_1 \simeq \frac{1}{4}e$ and $Q_2 \simeq \frac{1}{4}e$. The solutions for other vortices, such as $(0,\ 1)$ and $(1,\ -1)$ are non-self dual solutions. We find that ener-

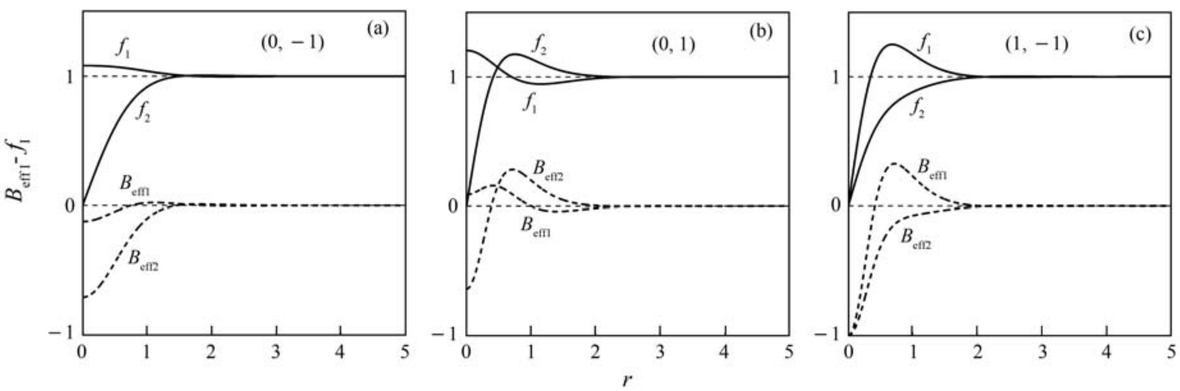

Fig. 1　Vortex solutions of type $(0,\ -1)$, $(0,\ 1)$ and $(1,\ -1)$ with $U = 2$ and $W = 2$. Solid lines show f_I and dashed lines denote $B_{\mathrm{eff}I}$.

gy of vortex $(0,\ 1)$ obtained through numerical integration, $E_{(0,\ 1)} \simeq 8.65$, which is much larger than that of the $(0,\ -1)$ type. And for type $(1,\ -1)$, $E_{(1,\ -1)} \simeq 13.86$. Vortex $(1,\ -1)$ will also decay.

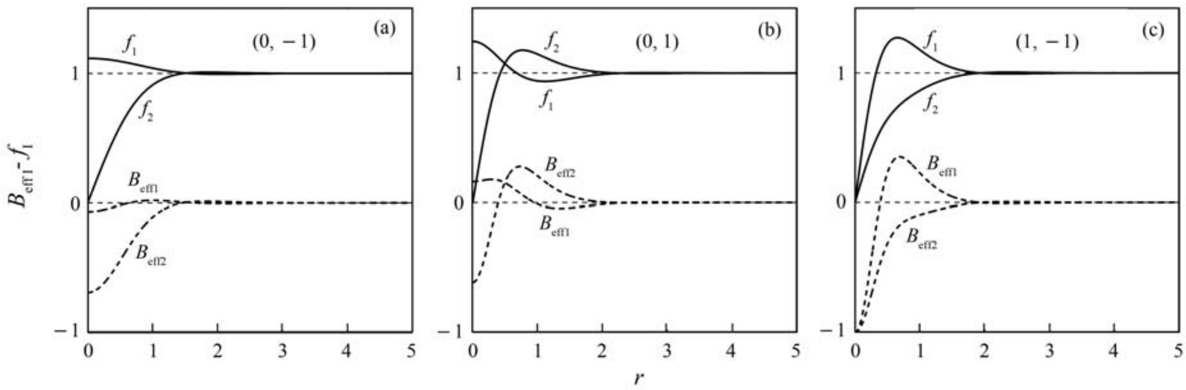

Fig. 2 Vortex solutions of type $(0,\ -1)$, $(0,\ 1)$ and $(1,\ -1)$ with $U = 2$ and $W = 0$.

Solid lines show f_I and dashed lines denote $B_{\text{eff}I}$.

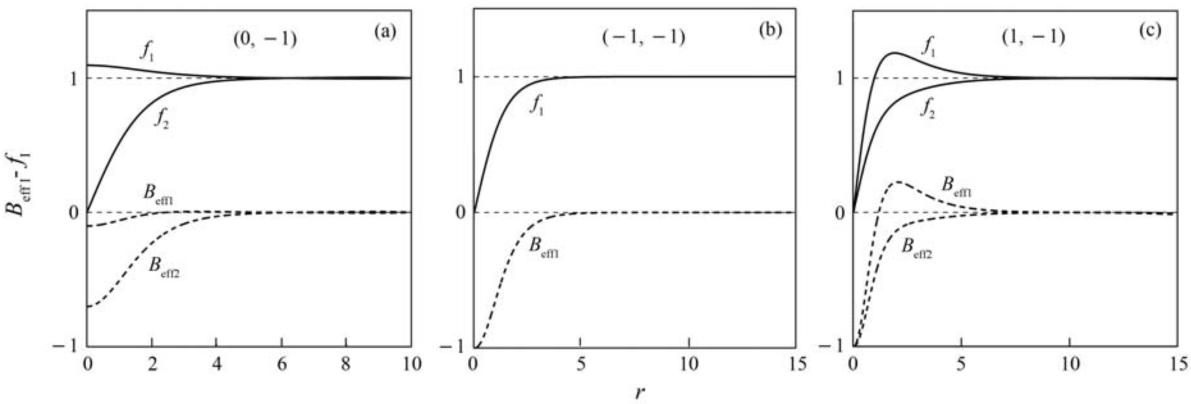

Fig. 3 The coupling constants take the critical values $U = 2\pi m/M$ and $W = 2\pi n/M$.

(a) and (b) are self-dual solutions, (c) is non-self dual solution. In the case of (b), fields in the two layers take the same value, the lines coincide with each other.

7 Summary

In this paper, we study the vortices in the bilayer quantum Hall systems at the filling factor $\upsilon = 1/2$. All the work is based on the ZHK model, which is an effective field theory with Chern-Simons gauge interactions. We take the assumption that the vortices are time independent and possess a cylindrical symmetry for simplicity. Consequently, the equations of motion could be radius dependent only. We rescale the variables to obtain the dimensionless Eqs. (15), (16) and (17), during which not only the fields but also the parameters are all dimensionless. These nonlinear coupled equations are solved through numerical methods. It is surprising that solutions satisfying asymptotical conditions in Eq. (23) do exist. We use these solutions to integrate the charges and effective fluxes, yielding the right fractional charge and flux unit. Vortices possess fractional charges in the two layers. $Q_1 \simeq -\dfrac{1}{8}e$ and $Q_2 \simeq \dfrac{3}{8}e$ for $(0,\ -1)$. $Q_1 \simeq \dfrac{1}{8}e$ and $Q_2 \simeq -\dfrac{3}{8}e$ for $(0,\ 1)$. The vortex of $(1,\ -1)$ is neutral, with $Q_1 \simeq -\dfrac{1}{2}e$ and $Q_2 \simeq \dfrac{1}{2}e$. We analyze the numerical energy values of vortices of the type $(0,\ -1)$, $(0,\ 1)$ and $(1,\ -1)$, finding that $(1,\ -1)$ will decay to $(0,\ -1)$ and $(1,\ 0)$. Additionally, we also analyze the form of static energy, finding that the self-dual solutions exist when the strength of self-interactions take the values $U = 2\pi m/M$ and $W = 2\pi n/M$. The vortices of type $(0,\ -1)$ and $(-1,\ -1)$ are indeed the self-dual solutions, the energy of which coincide with our analysis in Eq. (32).

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双层量子霍尔系统在填充因子$\upsilon = 1/2$态的涡流解

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摘要： 以双层系统的ZHK模型为基础，研究了双层量子霍尔系统在朗道填充因子取$\upsilon = 1/2$这种状态的静态涡流解。ZHK模型是一种包含Chern-Simons规范相互作用的有效理论。为了简便，假定涡流具有柱对称的结构，随后写出了无量纲的非线性运动方程组，并分析了解的渐进行为。另外，在自对偶条件下，确定了自耦合常数的形式，并写出了关于密度的自对偶方程。最后，使用数值方法找到了类型分别为(0, 1)，(0, −1)，(1, −1) 和(−1, −1) 的涡流解。发现拓扑数为(1, −1)的涡流是不稳定的,它会衰变为(1, 0)和(0, −1)两种涡流。数值结果表明，拓扑数为(0, −1)和(−1, −1)的涡流确实是自对偶涡流解。

关键词： 双层量子霍尔系统；Chern-Simons规范作用；自对偶条件；涡流

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