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Efimov States in Three-body Systems*

CHEN Shuang[1,2], REN Zhong-zhou[1,2,3,#], XU Chang[1,2]

(1 *Department of Physics, Nanjing University, Nanjing* 210008, *China*;
2 *Joint Center for Particle, Nuclear Physics and Cosmology, Nanjing University and Purple Mountain Observatory, Chinese Academy of Sciences Nanjing* 210008, *China*;
3 *Center of Theoretical Nuclear Physics, National Laboratory of Heavy Ion Research Facitity in Lanzhou, Lanzhou* 730000, *China*)

Abstract: We studied the Efimov effect in a three-body system by solving the Faddeev equations. Different models and interactions were considered. The occurrence of Efimov states was discussed. The possible Efimov state was clearly presented and its properties were investigated.

Key words: Efimov effect; Faddeev equation; three-body system

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1 Introduction

The Efimov effect[1] was first theoretically pointed out by Efimov that a number of weakly bound states arise in quantum three-body system when at least two of the binary subsystems have extremely large scattering length or bound states at nearly zero energy. Much effort have been made in finding and explaining these exotic bound excited states since then. Three-body models, solved by variational method[2,3], Faddeev calculation[4,5] and so on, have been successfully applied to study this effect. Evidence for Efimov quantum states in an ultracold gas of caesium atoms was firstly reported in 2006[6]. In nuclear physics, the weakly bound halo nuclei which can be described by a core+n+n model may have this kind of states[5,7—9]. The variational method has been successfully used to calculate the ground-state energies of quantum three-body system[2,3], but we know that this method does not apply to the calculation of the excited states. And due to the uncertainty of the interactions, the studies still have strong model dependence and the investigation of the universal aspect of this problem is in need. In this work we search for the Efimov states in three-body system by solving the Faddeev equations. Different models and interactions are considered in our study. We try to give some unified analysis of the properties by comparing ours with the previous results.

2 Method

We consider a system with three particles which are denoted as 1, 2 and 3. 2 and 3 are identical. Particle masses are $m_1 m$, $m_2 m$, $m_2 m$, respectively. m is nucleon mass. m_1, m_2 are integers. Two methods for solving this three-body system are involved in this work. One is the variational method[2]

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Biography: Chen Shuang(1984－), Male(Han Nationality), Songzi, Hubei, Postgraduate, working on theoretical nuclear physics, E-mail: shuangphy@gmail.com

Corresponding author: REN Zhong-zhou, E-mail: zren@nju.edu.cn

$$0=\delta\langle\Psi|H-E|\Psi\rangle,$$
$$\Psi=\Phi(R),\ R(\eta)=\frac{1}{2}(r_1+r_2+\eta r_3), \quad (1)$$

where r_1, r_2 and r_3 are the distances of 1-2, 1-3, and 2-3. $\Phi(R)$ is a s-state trial wave function[10] for three-body systems. η is a scaling parameter.

Another method for solving the Faddeev equations is

$$0=(T_i-E)\Psi_i+\sum_{i=1}^{3}V_i\Psi,\ (i=1,2,3)$$
$$\Psi=\Psi_1+\Psi_2+\Psi_3. \quad (2)$$

The hyperspherical expansion can be efficiently used in the three-body problem with short-range interactions[11]. The Hyperspherical decomposition of the wave function is used

$$\Psi^{JM}=\rho^{-5/2}\sum_{i,\Omega_i}\sum_{K_i}\chi^{L_iS_iIj_iJ}_{i,K_il_{xi}l_{yi}}(\rho)\varphi^{l_{xi}l_{yi}}_{i,K_i}(\theta_i)\times$$
$$\{[(Y_{l_{xi}}(x_i)\otimes Y_{l_{yi}}(y_i))_{L_i}\otimes(\chi_{s_j}\otimes\chi_{s_k})_{S_{xi}}]_{j_i}\otimes\chi_{s_i}\}_{JM}, \quad (3)$$

with the abbreviation $\Omega_i\equiv\{(l_{xi},l_{yi})L_i,(s_j,s_k)S_{xi}\}\times j_i$, and

$$\varphi^{l_{xi}l_{yi}}_{i,K_i}(\theta_i)=N^{l_{xi}l_{yi}}_{K_i}(\sin\theta_i)^{l_{xi}}(\cos\theta_i)^{l_{yi}}\times P^{l_{xi}+1/2,\,l_{yi}+1/2}_{(K_i-l_{xi}-l_{yi})/2}(\cos(2\theta_i)).$$

Here $(\rho,\ \theta_i)$ are the hyperspherical coordinates (hyper-radius $\rho=\sqrt{x_i^2+y_i^2}$ and hyper-angle $\theta_i=\arctan(x_i/y_i)$). $(x_i,\ y_i)$ are the corresponding Jacobi coordinates. For a given Jacobi set (x_i, y_i), we define the associate orbital angular momenta (l_{xi}, l_{yi}), the spins of the three particles s_i, s_j, s_k, the total angular momentum J and its projection M. The indexes i, j, k run through (1, 2, 3) in circular order. $P^{l_{xi}+1/2,\,l_{yi}+1/2}_{(K_i-l_{xi}-l_{yi})/2}$ is the Jacobi polynomial. $N^{l_{xi}l_{yi}}_{K_i}$ is a normalization coefficient and K_i is the hyper-angular-momentum. Using this expansion in the Faddeev equations, one can convert the two-dimensional partial differential equations (2) into a set of coupled one-dimensional equations

$$-\frac{\hbar}{2m}\left[\frac{d^2}{d\rho^2}+\frac{(K_i+3/2)(K_i+5/2)}{\rho^2}-E\right]\chi^{L_iS_iIj_iJ}_{i,K_il_{xi}l_{yi}}(\rho)+\sum_{j\alpha_jK_j}V^{ij}_{\Omega_iK_i,\Omega_jK_j}(\rho)\chi^{L_iS_iIj_iJ}_{i,K_il_{xi}l_{yi}}(\rho)=0, \quad (4)$$

where

$$V^{ij}_{\Omega_iK_i,\Omega_jK_j}(\rho)=\langle\varphi^{l_{xj}l_{yj}}_{j,K_j}(\theta_j)|\hat{V}_{ij}|\varphi^{l_{xi}l_{yi}}_{i,K_i}(\theta_i)\rangle.$$

The Laguerre polynomial expansion is used to solve these coupled equations[11].

3 Numerical Results and Discussions

At first, we assume that the three particles are spinless and take the two-body interactions as the s-wave potentials with Yukawa form[2]

$$V(r_3)=(-s)147.585(1/m_2)b^{-2}\times\frac{b}{r_3}\exp\left(-2.1196\frac{r_3}{b}\right),$$
$$U(r_{1,2})=(-s_c)\frac{m_1+m_2}{2m_1m_2}147.585b_c^{-2}\times\frac{b_c}{r_{1,2}}\exp\left(-2.1196\frac{r_{1,2}}{b_c}\right), \quad (5)$$

where $s(s_c)$ and $b(b_c)$ are parameters of the potential well depth and the force range, respectively. When $s(s_c)\leqslant 1$, the scattering length is negative and there is no bound state for the two-body subsystems[2]. The Faddeev equations are numerically solved to obtain the three-body binding properties. Considering the accuracy of our calculation, we show the convergence of ground-state energy in the hyperspherical expansion(see Fig. 1).

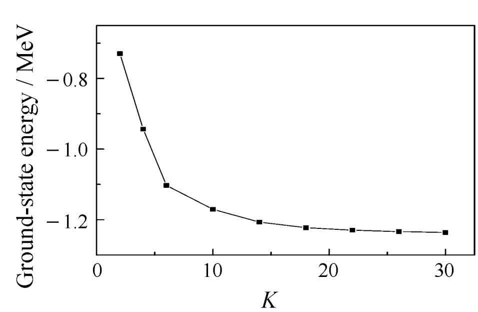

Fig. 1 Convergence of ground-state energy in Faddeev calculation: the three-body energy as a function of the truncated hyper-angular-momentum K. The hyper-radial wavefunction is expanded by 50 Laguerre polynomials. The hyper-radius ρ is calculated up to 600 fm. Yukawa potential parameters are taken as $s=s_c=1.0$ MeV, $b=b_c=5.0$ fm. $m_1=m_2=1$ are used as particle mass.

According to the results shown in Fig. 1, we use $K_{max}=32$ as a typical truncation in the following calculation in order to obtain correct binding energies. In Table. 1 we list the results for a series of three-body systems and compare them with those in Ref. [2]. As we can see, the ground and excited state energies calculated by Faddeev equations are lower than those calculated by the variational method. It's obvious that the Faddeev calculation is more precise if the convergence of the binding energies is ensured. Both the Faddeev and the variational calculation[2] give no bound excited state for the systems with three identical particles or with two light particles and a heavy particle if there is no bound state for two-body subsystems. We also try to search for the bound excited states for these systems by changing the mass ratio of particles but we do not find them. The bound excited states arise when we increase the potential strength s_c to a critical value(see row 2 and 4) which leads to a weakly bound state for the two-body subsystems. The Borromean CNN(Core-Nucleon-Nucleon) systems are not the candidates for searching Efimov states according to these results. For the systems with two heavy particles and a light particle, the results of both methods agree with the fact that such systems may have bound excited states although there is no bound subsystem. These bound excited states were discussed by Ren as Efimov states[2].

Table 1 Numerical results for three-body systems with Yukawa potentials($b=b_c=5.0$ fm). E_0 and E_1 are energies of the ground states and the first excited states, respectively. 'Faddeev' denotes the result of this work. 'Ren' is taken from Ref. [2].

m_1	m_2	s_c	Faddeev		Ren	
			E_0/MeV	E_1/MeV	E_0/MeV	E_1/MeV
1	1	1.00	−1.25	>0	−1.23	>0
1	1	1.20	−3.05	-0.79×10^{-2}		
1000	1	1.00	−1.00	>0	−0.99	>0
1000	1	1.31	−2.86	-0.60×10^{-2}		
1	100	1.00	−1.34	−0.15	−1.28	>0
1	1000	1.00	−1.83	−0.31	−1.78	-2.01×10^{-2}
1	10000	1.00	−2.08	−0.54	−2.06	-5.32×10^{-2}
1	100000	1.00	−2.18	−0.75	−2.17	-7.08×10^{-2}

According to the above calculation, one can search for Efimov states in the weakly bound three-body systems with two heavy particles and a light particle. In nuclear physics, one can hardly find this kind of systems. The three-body systems with two light particles and a heavy particle are the candidates if there are bound for core-n subsystems. In Fig. 2 we show the results for a series of CNN systems with $m_2=1$, $m_1=1-40$. The potential strength s_c is limited to the values which just allow the first excited state to arise in each group of calculation. $s=0.949$ MeV, $b=2.06$ fm are used to reproduce the low energy properties of NN interaction[2]. It is interesting to find that the core-n binding energy is nearly parameter-independent at this threshold. So we can approximately select the candidates in nuclear three-body systems on condition that the core-n subsystem should have a binding s state below about 0.45 MeV. According to the table given in Ref. [12], one can find that very few nuclei satisfy the condition shown in Fig. 2 if we only consider the ground states(the weakly bound properties which satisfy the Efimov condition may exist in the excited states of both stable and unstable nuclei). We can list a few nuclei such as ^{12,14}Be, ^{20}C. There are low-lying $1/2^+$

states which may exist in the core-n subsystems of these neutron rich nuclei. Many other works also selected out these nuclei[5] by different methods. Among the candidate nuclei, ^{20}C attracts the most attention[8, 13—16].

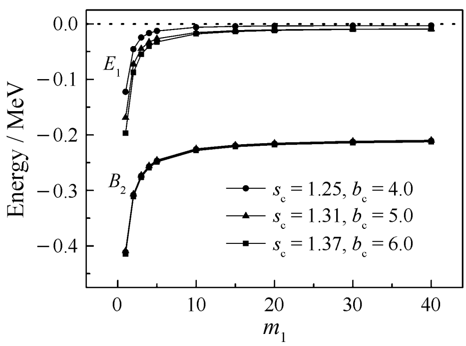

Fig. 2 Ground-state energies of the two-body systems(B_2) and the first excited state energies of the three-body systems(E_1) as functions of core mass(m_1). s=0.949 MeV, b=2.06 fm are used to reproduce the low energy properties of neutron-neutron interaction. s_c, b_c are selected to confirm that there are first bound excited states with minimal binding energy below about 0.01 MeV for the three-body systems. For B_2, the three lines can not be distinguished from each other.

Then we take more realistic calculation for three-body system with $m_1=18$, $m_2=1$. The spin 1/2 is considered for the two light particles. We use the spin-dependent Woods-Saxon potential

$$V_{\rm nc}=\frac{S}{1+{\rm e}^{(r_{\rm nc}-r_{\rm e})/a}}+S_{\rm so}\frac{l\cdot s}{ar_{\rm nc}}\frac{{\rm e}^{(r_{\rm nc}-r_{\rm e})/a}}{(1+{\rm e}^{(r_{\rm nc}-r_{\rm e})/a})}\ , \quad (6)$$

where n denotes the light particle and c denotes the heavy core particle. The interaction between the two light particles is used in the form[17] $V_{\rm nn}=(-31\ {\rm MeV})\exp[-r_{\rm nn}^2/(1.8\ {\rm fm})^2]$, which reproduces the measured low energy s-state neutron-neutron scattering length and effective range. The core spin is taken as zero. The core-n relative orbital angular momentum is calculated up to $2\hbar$. The results are shown in Fig. 3.

From the calculation we have $\langle\rho^2\rangle=732\ {\rm fm}^2$ at E_1=-0.1 MeV. The spatial extension of matter is far away from the potential range. It is no doubt that this bound excited state is an Efimov state. It is also confirmed in this calculation that there is no Efimov state when the core-n subsystem is unbound. Our result is consistent with those in Ref. [7], in which different method and interactions are used to solve the Faddeev equations.

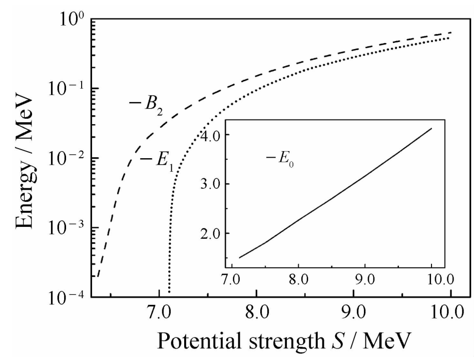

Fig. 3 The first bound excited energy E_1 and the core-n binding energy B_2 as a function of potential strength S(absolute value). The inset includes the three-body ground-state energy E_0. All the calculations are done for r_e = 2.80 fm, a=0.6 fm, S_{so} =−10.0 MeV.

4 Summary

In this paper we have studied the Efimov effect in quantum three-body systems. Firstly we conclude from the variational and Faddeev calculation that the Efimov effect may exist in the weakly bound three-body systems with two heavy particles and a light particle. Then the condition of occurrence of Efimov state in three-body systems with bound subsystems is investigated. Finally the Efimov state is clearly presented by realistic three-body calculation. More conditions, such as Coulomb interaction and finite core spin, should also be considered for further study. Since the long-range repulsive interactions hinder the Efimov effect[7], and the possible candidates mostly have spin-zero core, we only studied the three-body systems with spin-zero core and short-range interactions. According to our results and the previous works[2, 5, 7], the Efimov effect, as a critical phenomenon, may exist in three-body systems with short-range interactions. This conclusion is independent of the models and interactions used in these studies. The numerical distinction among the methods does not affect this qualitative properties.

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三体系统中的Efimov态*

陈　双[1,2]，任中洲[1,2,3,#]，许　昌[1,2]

(1 南京大学物理系，江苏 南京 210008;

2 南京大学-中国科学院紫金山天文台粒子-核-宇宙学联合研究中心，江苏 南京 210008;

3 兰州重离子加速器国家实验室原子核理论中心，甘肃 兰州 730000)

摘　要：通过求解Faddeev方程，研究了量子三体系统中的Efimov效应。改进了变分方法对于求解激发态的不足。在不同的两体作用下得到了三体系统中的Efimov态。讨论了在不同质量比的三体系统中出现Efimov态的条件。并由三体计算的结果分析了具有两个价中子的核系统在两体存在束缚态时可能存在的Efimov效应。

关 键 词：Efimov 效应；Faddeev 方程；三体系统

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通讯联系人：任中洲，E-mail：zren@nju.edu.cn