Article ID: 1007－4627(2007)01－0080－05

Expected Value of Finite Fission Chain Lengths of Pulse Reactors*

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Abstract: The average neutron population necessary for sponsoring a persistent fission chain in a multiplying system, is discussed. In the point reactor model, the probability function $\vartheta(n, t_0, t)$ of a source neutron at time t_0 leading to n neutrons at time t is dealt with. The non-linear partial differential equation for the probability generating function $G(z; t_0, t)$ is derived. By solving the equation, we have obtained an approximate analytic solution for a slightly prompt supercritical system. For the pulse reactor Godiva-Ⅱ, the mean value of finite fission chain lengths is estimated in this work and shows that the estimated value is reasonable for the experimental analysis.

Key words: pulse reactor; critical nuclear system; finite fission chain

CLC number: TL4　　**Document code**: A

1　Introduction

For a nuclear multiplying system of a weak neutron source with the reactive coefficient k, neglecting delayed neutrons, if a source neutron leads to a fission and sponsors a persistent fission chain, the length of the fission chain will present in stochastic fluctuation way, and the probability distribution of the chains will be strongly correlated to the value k. When $k<1$, the length of the fission chain will be finite, and, the probability of sponsoring a persistent fission chain is zero. When $k>1$, the length of the persistent fission chain may be finite or infinite, the probability mainly relies on the extent of $k>1$. i. e., even in a supercritical system, there is probability for a nuetron not to sponsor a persistent fission chain.

Although in the supercritical system with a permanent source S, the probability of sponsoring a persistent fission chain by the lasting injection of the source neutron in the time span from zero to infinity is unity, the randomicity of neutron transportation at micro lays often makes the time, length and probability, of persisitent fission chain, stochastic. So the time for sponsoring first persistent fission chain behaves in stochastic way.

In 1960, Wimett et al.[1] performed an experiment which clearly illustrated this effect. Using the Godiva-II burst assembly, 94 superprompt-critical bursts were initiated in which the average time-to-initiation was measured to be approximately 3 s, with a maximum of approximately 13 s.

Spriggs et al.[2] have ever pointed out that, at the time each burst was fired, the effective source strength that includes delayed neutron contribution, corresponded to approximately 1 000 n/s, thereforce, on an average, 3 000 source neutrons appeared in the system prior to the initiation of each burst. Since the probability of any neutron source causing an initial fission chains is approximately 40%, it follows that an average of 1 200

* Received date: 25 Apr. 2006; **Revised date**: 27 Jul. 2006

* **Foundation item**: Foundation of China Academy of Engineering Physics(2002y94)

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fission chains were initiated prior to the first persistent chain, of course, those 1 200 fission chains were finite length.

In the case of the burst that was delayed by 13 s, 13 000 source neutrons appeared in the system, initiating 5 200 prompt fission chains, which ultimately died-out before the first persistent prompt fission chain finally occurred. In fact, even when the system is superpromptly critical, most prompt fission chains are still desitined to die out, only occasionally a prompt fission chain will actually diverge.

In this paper a theoretical analysis to the Spriggs' conclusion of 1 200 fission chains with finite length is carried out. By calculating the averaged population of neutrons for sponsoring persistent fission chains in multiplication system and the mean value of finite fission chain lengths, and then solving the partial differential equation of generation function of the probability distribution, and then we could explain the physical quality of the probability distribution of burst waiting time on Godiva-II pulse reactor.

2 Necessity of Average Neutron Population to Sponsoring a Persistent Fission Chain

Consider a simple reactive system in which all neutrons behave identically.

Define W as the probability of a source neutron sponsoring a persistent fission chain, and then $(1-W)^n W$ expresses the probability of such a case, n neutrons in the system can not sponsor the persistent fission chain until the $(n+1)$th neutron is introduced into the system; $\bar{n}$ is the expected value of n neutrons introduced into the system to sponsor persistent fission chain.

Hence, we have

$$W+(1-W)W+(1-W)^2W+\cdots$$

$$(1-W)^n W+\cdots=W\sum_{n=0}^{\infty}(1-W)^n=1 \,. \quad (1)$$

So, in the system, the mean neutron population necessary for sponsoring a persistent fission chain is following:

$$\bar{n}=W\sum_{n=0}^{\infty}(n+1)(1-W)^n=\frac{1}{W} \,. \quad (2)$$

3 Equation of Probability and Approximate Analytic Solution

Define $\vartheta(n,t_0,t)$ as the probability that a neutron at time t_0 leads to n neutrons at time t in the system. Then $\vartheta(n,t_0,t)$ satisfies the following differential equation[3]:

$$\tau\frac{\mathrm{d}\vartheta(n,t_0,t)}{\mathrm{d}t}=$$

$$-n\vartheta(n,t_0,t)+(1-p)(n+1)\cdot$$

$$\vartheta(n+1,t_0,t)+p\sum_{\nu=0}^{\infty}P(\nu)(n+1-\nu)\cdot$$

$$\vartheta(n+1-\nu,t_0,t) \,, \quad (3)$$

with

$$\sum_{\nu=0}^{\infty}P(\nu)=1 \quad (4)$$

where τ is average neutron lifetime; p is the probability for each neutron to produce fission by absorption (with leakage included), and $P(\nu)$ the probability for each fission to emit ν neutrons. Eq. (3) described a Markov branching process.

$\vartheta(n,t_0,t)$ satisfies the initial condition $\vartheta(n,t_0,t)=\delta_{nl}$, with δ_{nl}, the Kronecker function, given by

$$\delta_{nl}=\begin{cases}1, & \text{for } n=l\\ 0, & \text{otherwise}\,.\end{cases}$$

Obviously $\vartheta(n,t_0,t)$ is normalized by the conservation condition.

$$\sum_{n=0}^{\infty}\vartheta(n,t_0,t)=1 \,. \quad (5)$$

We further introduce the generating function of probability distribution as follows:

$$G(z;t_0,t)=\sum_{n=0}^{\infty}\vartheta(n,t_0,t)z^n \,. \quad (6)$$

It is easily proven that

$$\vartheta(n,t_0,t)=\frac{1}{n!}\frac{\mathrm{d}G(z;t_0,t)}{\mathrm{d}z}\Big|_{z=0}\,. \tag{7}$$

Now multiplying Eq. (3) by z^n and summing over n, the partial differential equation of $G(z;t_0,t)$ is described as

$$\tau\frac{\partial}{\partial t}G(z;t_0,t)=\Big[p\Big(\sum_\nu P(\nu)z^\nu-z\Big)+(1-p)(1-z)\Big]\frac{\partial}{\partial z}G(z;t_0,t)\,, \tag{8}$$

where $G(z;t_0,t)$ satisfies the initial condition,

$$G(z;t_0,t)=z\,. \tag{9}$$

Then the normalization condition Eq. (5) may be re-expressed as:

$$G(1;t_0,t)=1\,. \tag{10}$$

The characteristic equation of the partial differential Eq. (7) is rewritten as follows:

$$\tau\frac{\mathrm{d}z}{\mathrm{d}t}=-p\Big(\sum_\nu P(\nu)z^\nu-z\Big)-(1-p)(1-z)\,. \tag{11}$$

According to the theory of the partial differential equation, the equation satisfied by the function $G(z;t_0,t)$ at t_0 is identical with the equation satisfied by the function $z(t)$ at t. Hence, we have

$$\tau\frac{\mathrm{d}}{\mathrm{d}t_0}G(z;t_0,t)=-p\Big(\sum_\nu P(\nu)[G(z;t_0,t)]^\nu-G(z;t_0,t)\Big)-(1-p)[1-G(z;t_0,t)]\,. \tag{12}$$

The initial condition Eq. (9) becomes a final condition

$$G(z;t,t)=z\,. \tag{13}$$

For more convenience we define a function $g(z;t_0,t)$ as

$$g(z;t_0,t)=1-G(z;t_0,t)\,. \tag{14}$$

By substituting Eq. (14) into Eq. (12), we get

$$\tau\frac{\mathrm{d}}{\mathrm{d}t_0}g(z;t_0,t)=p\Big\{\sum_\nu P(\nu)[1-g(z;t_0,t)]^\nu-1\Big\}+g(z;t_0,t)\,. \tag{15}$$

The final condition Eq. (13) becomes

$$g(z;t,t)=1-z\,, \tag{16}$$

with the normalizing condition

$$g(1;t_0,t)=0\,. \tag{17}$$

Suppose that the system is slightly supercritical, the generating functions may be very small. So the terms $(1-g(z;t_0,t))^\nu$ with $\nu\geqslant 3$ can be dropped, we obtain

$$(1-g)^\nu\approx 1-\nu g+\frac{1}{2}\nu(\nu-1)g^2\,. \tag{18}$$

Substituting the Eq. (18) into Eq. (15) and using condition $\sum_{\nu=0}P(\nu)=1$, it yields

$$\frac{\mathrm{d}}{\mathrm{d}t_0}g(z;t_0,t)=-\alpha g(z;t_0,t)+\frac{1}{2}p\,\overline{\nu(\nu-1)}g^2(z;t_0,t)\frac{1}{\tau} \tag{19}$$

with

$$\alpha=\frac{1}{\tau}(k-1),\ k=\bar{\nu}p,\ \bar{\nu}=\sum_\nu P(\nu)\nu\,,$$

$$\overline{\nu(\nu-1)}=\sum_\nu P(\nu)\nu(\nu-1)=\Gamma_2\bar{\nu}^2\,.$$

The Eq. (15) can be re-expressed as a linear differential equation for $1/g(z;t_0,t)$

$$\frac{\mathrm{d}}{\mathrm{d}t_0}\frac{1}{g(z;t_0,t)}=\frac{\alpha}{g(z;t_0,t)}-\frac{\Gamma_2\bar{\nu}^2}{2}p\frac{1}{\tau}\,. \tag{20}$$

Since Eq. (20) satisfies the final condition $g(z;t,t)=1-z$, the solution $g(z;t_0,t)$, the probability of a persistent chain reaction caused by a neutron injected into the system, of Eq. (20) can be obtained as

$$g(z;t_0,t)=\frac{\mathrm{e}^{\alpha(t-t_0)}}{\frac{\Gamma_2\bar{\nu}^2}{2}p\frac{1}{\tau\alpha}(\mathrm{e}^{\alpha(t-t_0)}-1)+\frac{1}{1-z}}\,. \tag{21}$$

Letting

$$a=\mathrm{e}^{\alpha(t-t_0)} \tag{22}$$

and

$$b = \frac{\Gamma_2 \bar{\nu}^2}{2} p \frac{1}{\tau\alpha} (e^{\alpha(t-t_0)} - 1) , \tag{23}$$

we further have

$$g(z; t_0, t) = \frac{a}{b} \frac{z-1}{z - \left(1 + \frac{1}{b}\right)}$$

$$= \frac{a}{1+b} \left[1 - \frac{1}{b} \sum_{n=1}^{\infty} \frac{z^n}{\left(1 + \frac{1}{b}\right)^n} \right] . \tag{24}$$

Substituting this last expression into Eq. (7), it gives

$$\vartheta(0, t_0, t) = 1 - \frac{a}{1+b} , \tag{25}$$

$$\vartheta(n, t_0, t) = 1 - \frac{a}{(1+b)b} \left(1 + \frac{1}{b}\right)^{-n} ,$$

$$n \neq 0 . \tag{26}$$

From $\alpha = (k-1)/\tau$, and the definition of reactivity $\rho = (k-1)/k$, Eq. (23) can be written as

$$b = \frac{\Gamma_2 \bar{\nu}^2}{2\rho} (e^{\frac{1}{\tau}(k-1)(t-t_0)} - 1) . \tag{27}$$

If the above mentioned W, namely the probability of a source neutron sponsoring a persisting chain reaction, could be regarded as the non-death probability, we should have

$$W = 1 - \lim_{t \to \infty} \vartheta(0, t_0, t)$$

$$= 1 - \lim_{t \to \infty} \left(1 - \frac{a}{1+b}\right)$$

$$= \lim_{t \to \infty} \left(\frac{a}{1+b}\right) = \frac{2}{\Gamma_2 \bar{\nu}} \rho , \tag{28}$$

and so Eq. (27) becomes

$$b = \frac{1}{W} (e^{\frac{1}{\tau}(k-1)(t-t_0)} - 1) . \tag{29}$$

When b is large enough, $\vartheta(n, t_0, t)$ can be made of continuum on the variable n, and written in a simple form as

$$\vartheta(n, t_0, t) = \frac{b}{(1+b)b} e^{-\frac{n}{b}} , \quad n \neq 0 \tag{30}$$

with the normalization relation

$$\vartheta(0, t_0, t) + \int_0^{\infty} \frac{a}{(1+b)b} e^{-\frac{n}{b}} dn = 1 , \tag{31}$$

and with the relation of the averaged neutron population:

$$\langle \bar{n} \rangle = \int_0^{\infty} n \frac{a}{(1+b)b} e^{-\frac{n}{b}} dn \approx a = e^{\alpha(t-t_0)} . \tag{32}$$

4 Mean Value of Finite Fission Chain Lengths of Godiva-Ⅱ Reactor

The formula (30) is just the analytical solution of the probability to a slightly prompt supercritical system of a point reactor model with its geometry independent of time for $n \geqslant 0$ and $t > t_0$.

Let t_1 be the time when the first persistent fission chain is sponsored, and let zero point be the time when a system with a step increasing reactivity becomes prompt critical. Hence, a source neutron introduced into the system at t_0 always produces a finite fission chain, when $t_0 < t_1$.

Let $n(t' = t - t_o)$ represent the time t distribution of neutron population from a source neutron injected into the system at t_0. For the given t', we always assume that the system has not been sponsored until the time t', hence the neutron number satisfies $0 \leqslant n \leqslant \bar{n}$, where $\bar{n}$ is equals to $1/W$ by Eq. (2). Therefore, the expected value of neutron population at the time t' is

$$\langle n(t') \rangle = \sum_{n=0}^{\frac{1}{W}} n \vartheta(n, 0, t') . \tag{33}$$

Then we could estimate the mean value of a finite fission chain lengths as follows

$$\langle n_f \rangle = \Sigma_f v \int_0^{\infty} dt' \sum_{n=0}^{\frac{1}{W}} \vartheta(n, 0, t') , \tag{34}$$

where Σ_f is the macroscopic fission cross section, and v the neutron speed. By substituting Eq. (30) into Eq. (34), the expected value of a finite fission chain lengths is rewritten as

$$\langle n_f \rangle = \Sigma_f v \int_0^{\infty} dt' \int_0^{\frac{1}{W}} n \frac{a}{(1+b)b} e^{-\frac{n}{b}} dn . \tag{35}$$

By using Eqs. (22), (23) and (29) and relation $k/\tau = \bar{\nu} v \Sigma_f$, we obtain

$$\langle n_{\rm f}\rangle = \frac{1}{\bar{\nu}\rho}\int_0^{\infty} {\rm d}e^{\alpha t'}\left[1 - e^{\frac{1}{\alpha t'-1}}\left(1+\frac{1}{\alpha\, t'-1}\right)\right]. \quad (36)$$

Let $y=1/(e^{\alpha t'}-1)$, thus $e^{\alpha t'}=(1/y)+1$. By completing the above integral, we obtain finally an estimated formula

$$\langle n_{\rm f}\rangle = \frac{1}{\bar{\nu}\rho}. \quad (37)$$

By substituting the parameters of the Godiva-II reactor ($\bar{\nu}=2.59$, $\rho(\$)=0.05$, $\beta_{\rm eff}=0.006\ 9$, $\Gamma_2=0.795$)[4] into this last formula, one can easily estimate the mean value of the Godiva-II reactor $\langle n_{\rm f}\rangle=1\ 119$.

This result seems to be quite close to that of the Spriggs' conclusion on experiment.

5 Conclusions

We have theoretically estimated that the mean value, $\langle n_{\rm f}\rangle=1\ 119$, of finite fission chain lengths of Godiva-Ⅱ. In the case of reactivity $\rho\ll 1$, $\langle n_{\rm f}\rangle$ is inversely proportional to $\bar{\nu}\rho$, i. e. $\langle n_f\rangle=1/\bar{\nu}\rho$. The conclusion may also be used for evaluating the chain length of a finite fission in other impulse reactor.

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脉冲堆有限裂变链长的数学期望值分析*

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摘　要：讨论了在一个增殖系统引发一个持续裂变链所需要的平均中子数。在点堆模型基础上，考虑了在 t_0 时刻系统引入一个源中子，在 t 时刻产生 n 个中子的概率 $\vartheta(n, t_0, t)$，推导了概率生成函数 $G(z; t_0, t)$ 所满足的偏微分方程，并得到了近似解。用近似解计算了 Godiva-II 脉冲堆的有限裂变链长数学期望值，有限裂变链期望值反比于脉冲堆的反应性。

关键词：脉冲堆；临界核系统；有限裂变链

* **收稿日期**：2006 - 04 - 25；**修改日期**：2006 - 07 - 26

* **基金项目**：中国工程物理研究院基金资助项目(2002y94)

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