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Vortex in Generalized Gross-Pitaevskii Theory*

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Abstract: We studied the topological structure of vortex in the Bose-Einstein condensation with a generalized Gross-Pitaevskii equation in (2+1)-dimensional space-time and 3-dimensional space, respectively. Such equation can be used in discussing Bose-Einstein condensates in heterogeneous and highly nonlinear systems. An explicit expression for the vortex velocity field as a function of the order parameter field is derived in terms of the Φ-mapping theory, and the topological structure of the velocity field is studied. At last, the branch conditions for generating, annihilating, crossing, splitting and merging of vortex in two kinds of Bose-Einstein systems are given.

Key words: Gross-Pitaevskii equation; topological current theory of Φ-mapping; bifurcation

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1 Introduction

Over the last few years a remarkable series of experiments on vapors of rubidiym[1], lithium[2], and sodium[3] have led to a renewed interest in the phenomenon of Bose-Einstein condensation (BEC). The quantum aspects of the vortex system in the Bose-Einstein condensation system is governed by a non-linear Schrödinger equation which is equivalent to the Gross-Pitaevskii equation[4]. However there is no satisfactory theory which describes the detail topological properties of the vortex, i. e. how one can define quantities like the density of vortices and an associated vortex velocity field.

In this paper we wish to explore the vortices of Bose-Einstein condensation system in detail. We will analyze some physically conceivable extensions of the usual Gross-Pitaevskii approximation. In particular, we will consider a arbitrary non-linear interactions depending explicitly on (space and time) position. By making use of the Φ-mapping theory, we will study the topological properties of the vortices in the (2+1)-dimensional Bose-Einstein condensation system governed by this generalized Gross-Pitaevskii equation. It will be shown that the vortices are generated from $\Phi=0$ and their topological charges are quantized under the condition $D(\Phi/x)\neq 0$. At the zero points of the order parameter Φ where the corresponding Jacobian determinant $D(\Phi/x)$ vanishes, the vortex topological current bifurcates and the vortices split at such point. These conditions give simple rules to consider the nonlinear behavior in all sort of Bose-Einstein condensation systems. Furthermore, we discuss the vortex-lines in 3-dimensional Bose-Einstein condensation and give the similar bifurcation results. They will help those experts to find a branch point and the concrete branch process in the neighborhood of it, and give a deep insight into Bose-Einstein condensation system.

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2 Vortex Topological Current with the Generalized Gross-Pitaevskii Equation in (2+1)-dimensional Space-time

We begin with an Landau-Ginzburg effective action of the form

$$S = \int \mathrm{d}t\mathrm{d}^3x\{\psi^*(\mathrm{i}\hbar\partial_t + \frac{\hbar^2}{2m}\nabla^2 - V_{\mathrm{ext}}(\boldsymbol{x}))\psi - \frac{1}{2}\lambda|\psi(t,\boldsymbol{x})|^4\},$$

where $\psi(t,\boldsymbol{x})$ is the wave function of the Bose-Einstein condensate. This action can be associated to a standard closed-form equation

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi(t,\boldsymbol{x}) = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\mathrm{ext}}(\boldsymbol{x}) + \lambda|\psi(t,\boldsymbol{x})|^2\right)\psi(t,\boldsymbol{x}), \tag{1}$$

which takes the name of the Gross-Pitaevskii equation (sometimes called the nonlinear Schrödinger equation). In order to discussing Bose-Einstein condensates in most general way we consider the generalized Gross-Pitaevskii equation

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi(t,\boldsymbol{x}) = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\mathrm{ext}}(\boldsymbol{x}) + f(\psi^*\psi)\right)\psi(t,\boldsymbol{x}). \tag{2}$$

Here we replace the quadratic $\lambda|\psi(t,\boldsymbol{x})|^2$ by an arbitrary nonlinearity $f(\psi^*\psi)=f(|\psi|^2)$. We note in particular that for two-dimensional systems Kolomeisky et al[5] have argued that in many experimentally interesting cases the nonlinearity will be cubic or even logarithmic in $|\psi|^2$. At the same time we permit the nonlinearity function and the confining potential to be explicitly space and time dependent, i.e. $f=f(\boldsymbol{x},t,[\psi^*\psi])$ and $V_{\mathrm{ext}}=V_{\mathrm{ext}}(t,\boldsymbol{x})$. According to above equation, one can construct the vortex current with the condensed wave function ψ

$$j = \frac{m}{\hbar}\nabla\times V,$$

where

$$V = -\frac{\mathrm{i}\hbar}{2m}\frac{\psi^*\nabla\psi - \psi\nabla\psi^*}{|\psi|^2},$$

is just the current velocity obtained from Eq. (2). Here one can notice that we didn't use the usual expression

$$\Psi = R\mathrm{e}^{\mathrm{i}\Theta}. \tag{3}$$

The reason is that with the expression (3) one will get the velocity

$$V = -\frac{\mathrm{i}\hbar}{m}\nabla\Theta,$$

then the vorticity is always trivial, i.e.

$$\nabla\times V = 0. \tag{4}$$

But in general $\nabla\times V$ can be non-zero at a singular line, which is found by Onsager and Feynman[6] and no exalt solution is given for many years. In the following we will discuss this question in detail.

It is well known that the condensed wave function ψ can be looked upon as a section of a complex line bundle with base manifold M (in this paper $M=R^2\odot R$). Denoting the condensed wave function ψ as

$$\psi(\boldsymbol{x}) = \Phi^1(\boldsymbol{x}) + \mathrm{i}\Phi^2(\boldsymbol{x}),$$

where $\Phi^1(\boldsymbol{x})$ and $\Phi^2(\boldsymbol{x})$ are two components of a two-dimensional vector field

$$\Phi = (\Phi^1,\Phi^2)$$

in the (2+1)-dimensional space-time. The current can be given

$$j^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\epsilon_{ab}\partial_\nu n^a\partial_\lambda n^b, \quad \mu,\nu,\lambda = 0,1,2, \tag{5}$$

where n^a is the two-dimensional unit vector field of the complex scalar field:

$$n^a = \frac{\Phi^a}{\|\Phi\|}, \quad \|\Phi\|^2 = \Phi^a\Phi^a, \ a = 1,2. \tag{6}$$

It is clear that the topological current is identically conserved, i.e.

$$\partial_\mu j^\mu = 0. \tag{7}$$

Substituting (5) into (6) and considering that

$$\partial_\mu n^a = \frac{\partial_\mu\Phi^a}{\|\Phi\|} - \Phi^a\partial_\mu\left(\frac{1}{\|\Phi\|}\right),$$

we have

$$j^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\epsilon_{ab}\partial_\nu\Phi^c\partial_\lambda\Phi^b\frac{\partial}{\partial\Phi^c}\frac{\partial}{\partial\Phi^a}\ln(\|\Phi\|).$$

If we define the vector Jacobian as

$$\epsilon^{ab}D^{\mu}\left(\frac{\Phi}{x}\right)=\epsilon^{\mu\nu\lambda}\partial_{\nu}\Phi^{a}\partial_{\lambda}\Phi^{b}$$

and by virtue of the Laplacian relation in Φ space

$$\frac{\partial}{\partial\Phi^{a}}\frac{\partial}{\partial\Phi^{a}}\ln(\|\Phi\|)=2\pi\delta(\Phi),$$

we can get a important result

$$\nabla\times V=\frac{\hbar}{m}\delta(\Phi)D^{\mu}\left(\frac{\Phi}{x}\right). \quad (8)$$

which is the rigorous result of $\nabla\times V$. Furthermore, a δ-function like current can be obtained

$$j^{\mu}=D^{\mu}\left(\frac{\Phi}{x}\right)\delta(\Phi). \quad (9)$$

Thus we have the important relation between the topological current and the condensed wave function $\psi(x)$ in the Bose-Einstein condensation system.

From the Eq. (9), we can see that topological current j^{μ} does not vanish only at the zero points of Φ, i.e.

$$\Phi^{1}(x,y,t)=0, \quad \Phi^{2}(x,y,t)=0. \quad (10)$$

The solutions of Eqs. (10) can be generally expressed as

$$x=x_{l}(t), \quad y=y_{l}(t), \quad l=1,2,\cdots,N, \quad (11)$$

which represent N zero points $z_l(t)$ $(l=1,2,\ldots,N)$ or worldlines of N vortices in space-time. The location of lth vortex is determined bythe lth zero point $z_l(t)$.

According to the Φ-mapping topological current theory[7], one can prove that

$$\delta^{2}(\psi)=\sum_{l=1}^{N}\frac{\beta_{l}}{|D(\Phi/x)|_{z_l}}\delta^{2}(z-z_{l}), \quad (12)$$

where the positive integer β_l is called the Hopf index[8] of map $x->\Phi$. The meaning of β_l is that when the point z covers the neighborhood of the zero z_l once, the vector field Φ covers the corresponding region β_l times. Following the Φ mapping topological current theory, we can obtain the general velocity of the lth zero

$$v^{\mu}=\frac{dx_{l}^{\mu}}{dt}=\frac{D^{\mu}(\Phi/x)}{D(\Phi/x)}\Big|_{z_l}, \quad v^{0}=1$$

from which one can identify the vortices velocity field as

$$v^{\mu}=\frac{D^{\mu}(\Phi/x)}{D(\Phi/x)}, \quad \mu=1,2 \quad (13)$$

where it is assumed that the velocity field is used inside expressions multiplied by the vortices locating δ function. The expressions given by Eq. (13) for the velocity of vortices are useful because they avoid the problem of having to specify the position of vortices explicitly. The positions are implicitly determined by the zeros of condensate wave function.

Then the vortex three-current j^{μ} can be written as the form of the current and the density of the system of N classical point particles with topological charge $W_l=\beta_l\eta_l$ moving in the $(2+1)$-dimensional space-time

$$j=\sum_{l=1}^{N}W_{l}v_{l}\delta^{2}(z-z_{l}(t)),$$

$$\rho=j^{0}=\sum_{l=1}^{N}W_{l}\delta^{2}(z-z_{l}(t)). \quad (14)$$

where η_l is the Brouwer degree[8]:

$$\eta_{l}=\frac{D(\Phi/x)}{|D(\Phi/x)|}\Big|_{z_l}=\pm 1.$$

It is clear to see that Eq. (14) shows the movement of the vortices in space-time.

3 The Generation and Annihilation of Vortices

As being discussed before, the zeros of the condensate wave function Ψ play an important role in studying the vortices in the Gross-Pitaevskii theory of Bose-Einstein condensation system. Now, we begin studying the properties of the zero points (locations of vortices), in other words, the properties of the solutions of Eqs. (10). As we know before, if the Jacobian

$$D\left(\frac{\Phi}{x}\right)=\frac{\partial(\Phi^{1},\Phi^{2})}{\partial(x^{1},x^{2})}\neq 0, \quad (15)$$

we will have the isolated solutions (11) of Eqs. (10). However, when the condition (15) fails, the usual implicit function theorem is of no use. The above results (11) will charge in some way and will lead to the branch process. We denote one of the zero points as (t^*, z_l). If the Jacobian

$$D^1\left(\frac{\Phi}{x}\right)\Big|_{(t^*,z_l)} \neq 0 , \qquad (16)$$

we can use the Jacobian $D^1(\Phi/x)$ instead of $D(\Phi/x)$ for the purpose of using the implicit function theorem. Then we have an unique solution of Eqs. (10) in the neighborhood of the points (t^*, z_l)

$$t = t(x^1) , \quad x^2 = x^2(x^1) , \qquad (17)$$

with $t^* = t(z_l^1)$. And we call the critical points (t^*, z_l) the limit points. In the present case, it is easy to know that

$$\frac{dx^1}{dt}\Big|_{(t^*,z_l)} = \frac{D^1(\Phi/x)|_{(t^*,z_l)}}{D(\Phi/x)|_{(t^*,z_l)}} = \infty , \qquad (18)$$

i.e. $$\frac{dt}{dx^1}\Big|_{(t^*,z_l)} = 0 .$$

The Taylor expansion of the solution of Eq. (17) at the limit point (t^*, z_l) is[7]

$$t - t^* = \frac{1}{2}\frac{d^2t}{(dx^1)^2}\Big|_{(t^*,z_l)}(x^1 - z_l^1)^2 \qquad (19)$$

which is a parabola in the x^1-t plane. From Eq. (19), we can obtain two solutions $x_1^1(t)$ and $x_2^1(t)$, which give two branch solutions (worldlines of vortices) of Eqs. (10). If $[d^2t/(dx^1)^2]|_{(t^*,z_l)} > 0$, we have the branch solutions for $t > t^*$, otherwise, we have the branch solutions for $t < t^*$. These two cases are related to the origin and annihilation of vortices.

From Eq. (18), we obtain an important result that the velocity of vortices is infinite when they are annihilating or generating, which is gained only form the topology of the condensate wave function.

Since the topological charge of vortices is identically conserved (7), the topological charges of these two vortices must be opposite at the limit point, i.e.,

$$\beta_{l_1}\eta_{l_1} = -\beta_{l_2}\eta_{l_2} , \qquad (20)$$

which shows that $\beta_{l_1} = \beta_{l_2}$ and $\eta_{l_1} = -\eta_{l_2}$.

4 Bifurcation of Vortex Three-current

For a limit point, it also requires $D^1(\Phi/x)|_{(t^*,z_l)} \neq 0$. As to a bifurcation point, it must satisfy a more complex condition at the bifurcation point (t^*, z_l):

$$\begin{cases} D\left(\frac{\Phi}{x}\right)\Big|_{(t^*,z_l)} = 0 \\ D^1\left(\frac{\Phi}{x}\right)\Big|_{(t^*,z_l)} = 0 \end{cases} \qquad (21)$$

which will lead to an important fact that the function relationship between t and x^1 is not unique in the neighborhood of the bifurcation point (t^*, z_l). It is easy to see from equation

$$\frac{dx^1}{dt}\Big|_{(t^*,z_l)} = \frac{D^1(\Phi/x)|_{(t^*,z_l)}}{D(\Phi/x)|_{(t^*,z_l)}} \qquad (22)$$

which under the restrain (21) directly shows that the direction of the integral curve of Eq. (22) is indefinite, i.e., the velocity field of vortices is indefinite at the point (t^*, z_l). This is why the very point (t^*, z_l) is called a bifurcation point of the condensate wave function.

Next, we will find a simple way to search for the different directions of all branch curves (or velocity field of vortex) at the bifurcation point. Assume that the bifurcation point (t^*, z_l) has been found from Eqs. (10) and (21). The Taylor expansion of the solution of Eqs. (10) in the neighborhood of the bifurcation point (t^*, z_l) can be expressed as[7]

$$A(x^1 - z_l^1)^2 + 2B(x^1 - z_l^1)(t - t^*) + C(t - t^*)^2 = 0 , \qquad (23)$$

which leads to

$$A\left(\frac{dx^1}{dt}\right)^2 + 2B\frac{dx^1}{dt} + C = 0 , \qquad (24)$$

and

$$C\left(\frac{dt}{dx^1}\right)^2 + 2B\frac{dt}{dx^1} + A = 0 , \qquad (25)$$

where A, B and C are three parameters. The solutions of Eq. (24) or Eq. (25) give different directions of the branch curves (worldlines of vortices) at the bifurcation point.

The remainder component dx^2/dt can be given by

$$\frac{dx^2}{dt} = f_1^2\frac{dx^1}{dt} + f_t^2$$

where partial derivative coefficients f_1^2 and f_1^2 have been calculated[7]. From these relations we find that the values of dx^2/dt at the bifurcation point (t^*, z_l) are also possible different because (23) may give different values of dx^1/dt. The above solutions reveal the evolution of vortices. Besides the encountering of the vortices, i.e., two vortices encounter and then depart at the bifurcation point along different branch point, it may split into several vortices along different branch curves. On the contrary, several vortices can merge into one vortex at the bifurcation point. The identical conversation of the topological charge shows the sum of the topological charge of final vortices must be equal to that of the initial vortices at the bifurcation point, i.e.,

$$\sum_f \beta_{l_f}\eta_{l_f} = \sum_i \beta_{l_i}\eta_{l_i} ,$$

for fixed j. Furthermore, from above studies we see that the generation, annihilation and bifurcation of vortices are not gradual charges, but start at a critical value of arguments, i.e. a sudden charge.

5 Vortex Topological Current with the Generalized Gross-Pitaevskii Equation in 3-dimensional Space

In this section, we will discuss the vortex lines in the 3-dimensional space. Similarly we have the generalized Gross-Pitaevskii equation

$$i\hbar\frac{\partial}{\partial t}\psi(t,\boldsymbol{r}) = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{ext}(\boldsymbol{r}) + f(\psi^*\psi)\right)\psi(t,\boldsymbol{r}) . \quad (26)$$

where $\boldsymbol{r}=(x,y,z)$ is different from parameter $\boldsymbol{x}=(x,y)$ in the Section 2. With the same procedure the current velocity can be given from above equation

$$\boldsymbol{V} = -\frac{i\hbar}{2m}\frac{\psi^*\nabla\psi - \psi\nabla\psi^*}{|\psi|^2} ,$$

then we can give the intrinsic relation between condensate wave function and the vorticity

$$\nabla\times\boldsymbol{V} = \frac{\hbar}{m}\delta(\Phi^1)\delta(\Phi^2)\boldsymbol{D}\left(\frac{\Phi}{x}\right) , \quad (27)$$

where the vector Jacobians $\boldsymbol{D}(\Phi/x) = (D_x(\Phi/x), D_y(\Phi/x), D_z(\Phi/x))$ can be defined as

$$D_x\left(\frac{\Phi}{x}\right) = \det\begin{bmatrix}\partial_y\Phi^1 & \partial_z\Phi^1\\ \partial_y\Phi^2 & \partial_z\Phi^2\end{bmatrix} ,$$

$$D_y\left(\frac{\Phi}{x}\right) = \det\begin{bmatrix}\partial_z\Phi^1 & \partial_x\Phi^1\\ \partial_z\Phi^2 & \partial_x\Phi^2\end{bmatrix} ,$$

$$D_z\left(\frac{\Phi}{x}\right) = \det\begin{bmatrix}\partial_x\Phi^1 & \partial_y\Phi^1\\ \partial_x\Phi^2 & \partial_y\Phi^2\end{bmatrix} ,$$

$\psi=\Phi^1+i\Phi^2$ and $\boldsymbol{e}_k(k=1,2,3)$ are the base vectors in Cartesian coordinate system.

From Eq. (27), we see that the vorticity $\nabla\times\boldsymbol{V}$ is infinitely great at the zero points of the condensate wave function, i.e.,

$$\Phi^1(x,y,z) = 0 , \quad \Phi^2(x,y,z) = 0 . \quad (28)$$

When the vector Jacobian $\boldsymbol{D}(\Phi/x)\neq 0$, the solutions of Eq. (28) are generally expressed as

$$x = x_k(s) , \quad y = y_k(s) , \quad z = z_k(s) , \quad k = 1,2,\ldots,N . \quad (29)$$

which represent N isolated zero lines L_k $(k = 1, 2,\ldots,N)$ of the condensate wave function.

When the wave function ψ has some isolated zero lines (29), one can get the topological structure of the vortex lines:

$$\nabla\times\boldsymbol{V} = \frac{\hbar}{m}\sum_{k=1}^{N}\beta_k\eta_k\int_{L_k} d\boldsymbol{r}_k\delta^3(\boldsymbol{r}-\boldsymbol{r}_k) , \quad (30)$$

where the line integral is taken along the vortex-lines. The positive integer β_k is the Hopf index and $\eta_k=\pm 1$ is the Brouwer degree of Φ-mapping. It is obvious that Eq. (30) represents N isolated vortex lines of which the k-th vortex-lines carries charge $\beta_k\eta_k\hbar/m$.

Let Σ be an arbitrary surface and suppose that there are P vortex-lines passing through it. According to (27), one can prove that

$$\int_\Sigma \nabla\times\boldsymbol{V}\cdot d\boldsymbol{\sigma} = \sum_{l=1}^{P}\beta_l\eta_l ,$$

which confirms that $\nabla\times\boldsymbol{V}$ represents the line density of vortex-line. Thus, the line density of vortex-lines $\nabla\times\boldsymbol{V}$ can be expressed in terms of the condensate wave function (27).

6 Bifurcation of Vortex Lines in the 3-dimensional Space

As being discussed before, the zeros of the condensate wave function ψ play an important role in studying the vortex-lines in the Gross-Pitaevskii theory. Now, we begin studying the properties of the zero points (locations of vortex-lines), in other words, the properties of the solutions of Eqs. (28). As we know before, if the Jacobian $D(\Phi/x) \neq 0$, we will have the isolated solutions (29) of Eqs. (28). When the condition fails, the above results (29) will change in some way. It is interesting to discuss what will happen and what is the correspondence in physics when $D(\Phi/x) = 0$ at some zero points $r_k = (x_k^*, y_k^*, z_k^*)$ of ψ. When $D(\Phi/x) = 0$ at some zero points, $\delta(\Phi^1)\delta(\Phi^2)D(\Phi/x)$ is indefinite (for $\delta(\Phi^1)\delta(\Phi^2)$ is infinitely large at these points). So, the vorticity is indefinite at these points which are called the bifurcation points.

According to the Φ-mapping topological current theory, the Taylor expansion of the solution of Eqs. (28) in the neighborhood of the bifurcation point r_k^* can be generally expressed as[7]

$$A(x - x_k^*)^2 + 2B(x - x_k^*)(z - z_k^*) + C(z - z_k^*)^2 + 0(|x - x_k^*|^2 + |z - z_k^*|^2) = 0 , \quad (31)$$

which leads to

$$A\left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^2 + 2B\frac{\mathrm{d}x}{\mathrm{d}z} + C = 0 , \quad (32)$$

where A, B and C are three constants determined by the condensate wave function at the bifurcation point. The direction of vortex-lines at the bifurcation point is

$$\boldsymbol{m} = \left(\frac{\mathrm{d}x}{\mathrm{d}z}, \frac{\mathrm{d}y}{\mathrm{d}z}, 1\right) ,$$

where the second component $\mathrm{d}y/\mathrm{d}z$ of the direction vector $\boldsymbol{m}$ can be given by[7]: $\mathrm{d}y/\mathrm{d}z = f_1^2 \mathrm{d}x/\mathrm{d}z + f_3^2$, in which the partial derivative coefficients f_1^2 and f_3^2 are also determined by the condensate wave function at the bifurcation point. The solutions of Eq. (32) give the direction $\boldsymbol{m}$ of the zero lines of the condensate wave function. The number of vortex-lines passing through the bifurcation point is determined by the higher terms of the Taylor expansion (31).

In the case Eq. (32) gives two different directions of zero lines at the bifurcation point, $\delta(\Phi^1)\delta(\Phi^2)D(\Phi/x)$ is indefinite and then the vorticity is indefinite at the bifurcation point. In this case, vortex-lines with two different directions will pass through the bifurcation points, i. e., two vortex-lines intersect with two different directions at r_k^*. Thus one can see the interesting result that when vortex-lines cross, the vector Jacobian of the condensate wave function $D(\Phi/x)$ will vanish.

In the case Eq. (32) gives only one direction of zero lines at the bifurcation point, $\delta(\Phi^1)\delta(\Phi^2)D(\Phi/x)$ is definite and then vorticity is definite at the bifurcation point. In this case, vortex-lines with one direction will pass through the bifurcation points. This case also includes two other important situations. First, one vortex-line splits into two vortex-lines at the bifurcation point. Second, two vortex-lines merge into on vortex-line at the bifurcation point. One can see another interesting result: when vortex-lines split or merge, the vector Jacobian $D(\Phi/x)$ will vanish.

The above solution reveal the space bifurcation structure of the vortex-lines. Besides the intersection of vortex-lines, i. e. two vortex-lines intersect at the bifurcation point, splitting and mergence of flux-lines are also included. When a multicharged vortex-lines passes through the bifurcation point, it may split into two vortex-lines, moreover, two vortex-lines can merge into one vortex-lines at the bifurcation point. For the divergence of the line density of vortex-lines is zero, the sum of the topological charges of final vortex-line (s) must be equal to that of the initial vortex-line (s) at the bifurcation point, i. e., $\sum_f \beta_f \eta_f = \sum_i \beta_i \eta_i$. Now the topological structure of the vortex lines should be written as

$$\nabla \times \boldsymbol{V} = \begin{cases} \dfrac{\hbar}{m} \sum_i \beta_i \eta_i \int_{L_i} \mathrm{d}\boldsymbol{r}_i \delta^3(\boldsymbol{r} - \boldsymbol{r}_i) & z < z^*, \\ \dfrac{\hbar}{m} \sum_f \beta_f \eta_f \int_{L_f} \mathrm{d}\boldsymbol{r}_f \delta^3(\boldsymbol{r} - \boldsymbol{r}_f) & z > z^*, \end{cases}$$

which is another case ($\boldsymbol{D}(\Phi/x)=0$) of Eq. (27).

In the above discussions, we consider only the case that two vortex-lines pass through the bifurcation point. The configurations in which more vortex-lines pass through the bifurcation point are high-energy states and not stable states, and will fall apart as quirk as they form. However, we must point out that when several (more than two) vortex lines cross, split or merge at a certain time t^*, the branch conditions are still valid, that is, $\boldsymbol{D}(\Phi/x)|_{(t^*, r^*)}=0$, although these configurations are unstable.

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推广的 Gross-Pitaevskii 理论中的涡旋问题*

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摘　要：利用推广 Gross-Pitaevskii 方程，分别研究了(2+1)维时空和 3 维空间的 Bose-Einstein 凝聚体中涡旋的拓扑结构. 这一推广的方程能够被用于非均匀并且高度非线形的 Bose-Einstein 凝聚系统. 利用 Φ 映射拓扑流理论，给出了基于序参数的涡旋速度场，以及该速度场的拓扑结构. 最后，仔细地探讨了这两种 Bose-Einstein 系统中涡旋的各种分支条件.

关键词：Gross-Pitaevskii 方程；Φ 映射拓扑流理论；分岔

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