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# Nonexistence of Finite Positive Invariant Measure on Local Gauge Group $C^\infty(M,G)$

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**Abstract**: In is proved that on the local gauge group $C^\infty(M,G)$ ($M$ a compact manifold and $G$ a matrix Lie group) there does not exist a finite positive invariant measure.

**Key words**: Faddeev-Popov method; local gauge group; invariant measure

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Since Feynman[1] introduced path integral into quantum theories, the path integral (functional integral) has played an ever-increasing role in the study of quantum mechanics, especially quantum field theories (QFT). In 1960s Faddeev and Popov[2] successfully applied the path integral to the quantization of non-Abelian gauge theories, which proves to be difficult in the usual framework of canonical quantization. The F-P method itself has now become a commonly employed technique in field theories. But when applied it to some non-covariant gauges, problems may arise, on one hand. For example, in the Coulomb gauge, Christ and Lee[3] found that the Feynman rule should contain some additional terms which cannot be obtained through the F-P method. While in the axial gauge the F-P method itself cannot tell us how to correctly treat the pole in the gauge field propagator. On the other hand, the key point of the F-P method is constructing a gauge invariant functional of the gauge potential using the translational invariance of some measure on the local gauge group. In mathematics the existence of an invariant measure is only established for locally compact groups while the local gauge group is infinite dimensional and not locally compact. So here the measure problem deserve a careful study. In this paper we prove that on a general local gauge group $C^\infty(M,G)$ ($M$ a compact manifold, $G$ a matrix Lie group) there does not exist a finite translationally invariant positive measure.

The local gauge group $G_0=C^\infty(M,G)$ is endowed with the $C^\infty$-topology[4]. Let $\alpha: \boldsymbol{R}\to M$ be a smooth curve satisfying $\alpha'(0)\neq 0$. For any $\omega\in C^\infty(M,G)$, $\tilde{\omega}=\omega\circ\alpha$ is a smooth curve in the matrix group $G$. It is easily seen that $\tilde{\omega}(t)^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\omega}(t)|_{t=0}$ belongs to the Lie algebra $g$ of $G$. For $\xi=\sum_{i=1}^{n}u_i\xi_i\in g$ (where $\{\xi_1\cdots\xi_n\}$ is a basis of $g$) we define $f(\xi)=\exp(-\sum_{i=1}^{n}u_i^2)$ and consider the following continuous function on $C^\infty(M,G)$:

$$F[\omega]+f\left((\tilde{\omega}(t)^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\omega}(t)|_{t=0}\right). \qquad (1)$$

Let $\mu$ be a nonzero finite positive measure on the Borel $\sigma$-algebra of $G_0=C^\infty(M,G)$. We shall prove that $\mu$ cannot be (left or right) translationally invariant. Suppose it is not the case and $\mu$ is right translationally invariant, for example. Then we have the following equality ($F[\omega]$ is $\mu$-integrable):

$$\int_{G_0}D\mu(\omega)F[\omega]=\int_{G_0}D\mu(\omega)F[\omega\omega_0],\ \forall\ \omega_0\in G_0. \qquad (2)$$

From Eq(1) we have

$$\int_{G_0}D\mu(\omega)f\left(\tilde{\omega}(t)^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\omega}(t)|_{t=0}\right)$$

$$=\int_{G_0}f(\tilde{\omega}_0(0)^{-1}\tilde{\omega}(t)^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\omega}(t)|_{t=0}\tilde{\omega}_0(0)-$$

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$$\tilde{\omega}_0(t)^{-1}\frac{d}{dt}\tilde{\omega}_0(t)|_{t=0}),\ \forall\ \omega_i \in G_0. \qquad (3)$$

In Eq(3) we set $\omega_0$ to satisfy the condition $\tilde{\omega}_0(0)=I$ but arbitary otherwise. From the fact that $\frac{d}{dt}\tilde{\omega}_0(t)|_{t=0}$ can take on any value in the Lie algebra $g$ we have

$$\int_{G_0} D\mu(\omega) f\left(\tilde{\omega}(t)^{-1}\frac{d}{dt}\tilde{\omega}(t)|_{t=0}\right) = \int_{G_0} D\mu(\omega) f\left(\tilde{\omega}(t)^{-1}\frac{d}{dt}\tilde{\omega}(t)|_{t=0}+\xi\right), \quad \forall \xi \in g. \qquad (4)$$

Let us consider the function $f(\tilde{\omega}(t)^{-1}\frac{d}{dt}\tilde{\omega}(t)|_{t=0}+\xi)$ defined on $C^\infty(M,G)\times g$. This function is measurable (because it is continuous). We observe that ($m$ is the Lebesgue measure on $g$)

$$\int_{G_0} D\mu(\omega)\int_g dm(\xi)\,|f(\tilde{\omega}(t)^{-1}\frac{d}{dt}\tilde{\omega}(t)|_{t=0}+\xi)| = \int_{G_0} D\mu(\omega)\int_g dm(\xi) f(\tilde{\omega}(t)^{-1}\frac{d}{dt}\tilde{\omega}(t)|_{t=0}+\xi) = \int_{G_0} D\mu(\omega)\int_g dm(\xi) f(\xi) = \mu(G_0)\int_g dm(\xi) f(\xi) \qquad (5)$$

which shows that this iterated integral exists. Then according to Fubini theorem, both of the following two iterated integrals exist and are equal:

$$\int_{G_0} D\mu(\omega)\int_g dm(\xi) f\left(\tilde{\omega}(t)^{-1}\frac{d}{dt}\tilde{\omega}(t)|_{t=0}+\xi\right) = \int_g dm(\xi)\int_{G_0} D\mu(\omega) f(\tilde{\omega}(t)^{-1}\frac{d}{dt}\tilde{\omega}(t)|_{t=0}+\xi). \qquad (6)$$

The LHS of Eq(6) is $\mu(G_0)\int_g dm(\xi) f(\xi)$ while the RHS of Eq(6) is $+\infty$ according to Eq(4) (we have $\int_{G_0} D\mu(\omega) f(\tilde{\omega}(t)^{-1}\frac{d}{dt}\tilde{\omega}(t)|_{t=0})>0$). This is a contradiction, which shows that Eq(3) cannot be satisfied. Thus we have proved that $\mu$ cannot be right translationally invariant. By considering the function $f(\tilde{\omega}(t)\frac{d}{dt}\tilde{\omega}(t)^{-1}|_{t=0})$ we can similarly show that $\mu$ cannot be left translatioally invariant either. Thus we have proved that on the local gauge group $C^\infty(M,G)$ there does not exist a non-zero finite positive translationally invariant measure on its Borel $\sigma$-field.

In summary we have proved that on the local gauge group $C^\infty(M,G)$ there does not exist a finite positive invariant measure. It should be noted that the method provided in this paper is only appropriate for the study of finite measure on the group $C^\infty(M,G)$. From the consideration in this paper we cannot establish the existence or nonexistence of an invariant measure satisfying $\mu(C^\infty(M,G))=+\infty$.

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# 局域规范群 $C^\infty(M,G)$上不存在有限不变正测度


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**摘　要**：研究了对 Faddeev-Popov 方法有关键意义的局域规范群上不变测度的存在性问题. 证明了局域规范群 $C^\infty(M,G)$（$M$ 为一紧致流形，$G$ 为一矩阵李群）上不存在有限的平移不变的正测度.



**关键词**：Faddeev-Popov 方法；局域规范群；不变测度